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UDC 621.372.8.01

Radiation Conditions and Uniqueness Theorems for Open Waveguides*

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The correct formulation of boundary-value problems in the theory of open waveguides is considered.

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INTRODUCTION

The radiation conditions at infinity are an element of the foundation on which the solutions of exterior boundary-value problems for harmonic oscillations ($\sim e^{-ikct}$, $\mathrm{Im}\,k = 0$) are constructed. The physical meaning of the radiation principle was formulated in its time by Sommerfeld and consists of the absence of sources at infinity. To obtain a unique solution a different principle is also sometimes employed—the principle of limiting absorption: the solution is taken as the limit of a bounded solution as $\mathrm{Im}\,k \to +0$. In spite of the convenience of this principle the existence of this limit must be justified for each specific class of problems.

Sommerfeld derived the radiation conditions for free space containing bounded obstacles [1] (see also [2]). Sveshnikov derived an analogous result for an open waveguide in the form of a partial radiation condition for each of the characteristic waves, and he justified the principle of limiting absorption [3, 4]. Conditions of this type, as shown in [5, 6], can also be employed with complex k for analyzing exterior problems of the characteristic oscillations of obstacles.

The purpose of this paper is to derive the radiation condition for open waveguides (OW), as an extension of Sommerfeld's and Sveshnikov's condition to the case of unbounded space containing infinite bodies and surfaces, regular along some straight line. The method of deriving this condition follows from the general theory of partial differential equations [7, 8]: all possible forms of Green's function for the OW must be constructed, their asymptotic behavior at infinity must be studied, and the conditions that separate the unique form of Green's function satisfying the radiation principle must be formulated. In addition, it is of interest to prove the principle of limiting absorption.

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*Originally published in Radiotekhnika i elektronika, No. 12, 1988, pp. 2483-2491.

ISSN8756-6648/89/0007-0107$7.50/0

1. GREEN'S FUNCTIONS OF OPEN WAVEGUIDES

Consider a three-dimensional OW formed by elements of three types: dielectric rods $D \times z$, ideally conducting rods $M \times z$, and ideally conducting, open, infinitely thin surfaces $\partial M' \times z$. We shall assume that the contours of the transverse sections of the elements are closed curves ∂D, ∂M and the open curves $\partial M'$ can be multiply connected, but finite, and do not intersect and are quite smooth—for example, they have a continuous curvature (Fig. 1). We shall assume that all the media are nonmagnetic $\mu(\vec{r}) \equiv 1\ (\vec{r} = (r, \varphi))$ and the permittivity $\varepsilon(\vec{r})$ is a piecewise-differentiable function, equal to unity outside D; ∂D is the union of all lines where the permittivity is discontinuous. We denote by a the radius of the smallest circle containing the cross section of all elements of the OW, and the origin of coordinates is placed at the center of this circle; we also denote by $\partial W = \partial D \cup \partial M \cup \partial M'$ the collection of boundaries of elements in $\bar{W} = M \cup \partial W$.

The construction of the tensor Green's function for the OW $\hat{G}(\vec{R}, \vec{R}_0)$ reduces to searching for a pair of vector functions $\vec{G}^{e,m} = \{\vec{E}^{e,m}, \vec{H}^{e,m}\}$, such that

$$\begin{bmatrix} \vec{E}^{\alpha} \\ \vec{H}^{\alpha} \end{bmatrix} = \hat{G} \begin{bmatrix} \delta_{\alpha e} \vec{I}^{e} \\ \delta_{\alpha m} \vec{I}^{m} \end{bmatrix};$$

$$\hat{G} = \begin{bmatrix} \hat{G}_{ee}\, \hat{G}_{em} \\ \hat{G}_{me} \hat{G}_{mm} \end{bmatrix}, \quad \alpha = e, m, \tag{1}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and $|\vec{I}^{\alpha}| = 1$. The components of $\vec{G}^{\alpha}$ satisfy Maxwell's equations with a δ function on the right-hand side [9]:

$$\begin{aligned} &\operatorname{rot} \vec{E}^{\alpha} - ik\vec{H}^{\alpha} = -4\pi c^{-1} \delta_{\alpha m} \vec{J}^{m} \\ &\operatorname{rot} \vec{H}^{\alpha} + ik\varepsilon \vec{E}^{\alpha} = 4\pi c^{-1} \delta_{\alpha e} \vec{J}^{e}, \quad \vec{R}, \vec{R}_0 \in \mathbb{R}^3 \backslash (\bar{W} \times z), \quad \operatorname{Im} k = 0, \end{aligned} \tag{2}$$

where $\vec{J}^{\alpha} = \vec{I}^{\alpha} \delta(\vec{R} - \vec{R}_0)$ and $\alpha = e$ and m, and known conditions (the outer brackets denote the difference of limiting values)

$$[\vec{E}^{\alpha} \times \vec{n}]|_{(\partial M \cup \partial M') \times z} = 0; \quad [[\vec{E}^{\alpha} \times \vec{n}]]|_{\partial D \times z} = 0; \tag{3}$$
$$[[\vec{H}^{\alpha} \times \vec{n}]]|_{\partial D \times z} = 0,$$

$$\int_V (\operatorname{Re} \varepsilon |\vec{E}^{\alpha}|^2 + |\vec{H}^{\alpha}|^2)\, dv < \infty, \quad V \subset \mathbb{R}^3, \quad \alpha = e, m. \tag{4}$$

Based on general considerations we know that the solution of the exterior problem (2)-(4) without the additional condition at infinity ($R \to \infty$) will not be unique. However, since the conditions (3) are given on the boundary of the OW $\partial W \times z$, which also recedes to infinity, there are not enough grounds for using Sommerfeld's condition as usually done in classical diffraction

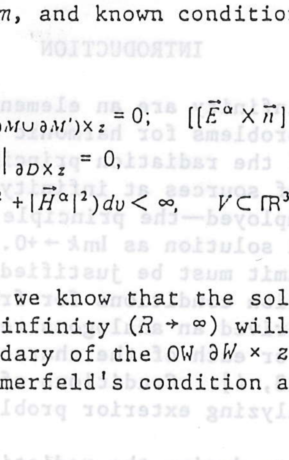

Fig. 1

problems. For this reason, it is first necessary to determine which solutions of (2)-(4) can exist.

We shall seek the functions $\vec{G}^{e,m}(\vec{R})$ among the class of functions which increase as $|z| \to \infty$ no faster than some power of $|z|$. In this case the generalized Fourier transformation can be applied to the solution of problems (2)-(4), so that

$$\vec{G}^{e,m}(\vec{R}, \vec{R}_0; k) = (2\pi)^{-1} \int_{-\infty}^{\infty} \vec{g}^{e,m}(\vec{r}, \vec{r}_0; k, h) e^{ih(z-z_0)} dh, \tag{5}$$

and the Fourier transforms of Green's functions $\vec{g}^{e,m} = \{\vec{u}^{e,m}, \vec{v}^{e,m}\}$ exist at least as generalized functions of the parameter h (IM $h = 0$), i.e., they can have singularities on the integration contour which are circumscribed along semicircles with infinitely small radius. We have the following boundary-value problem in the plane of the transverse cross section of the OW for the Fourier transforms $\vec{g}^{e,m}$:

$$\begin{aligned} &\operatorname{rot} \vec{u}^{\alpha} + ih[\vec{z}^{0} \times \vec{u}^{\alpha}] - ik\vec{v}^{\alpha} = -4\pi c^{-1} \delta_{\alpha m} \vec{j}^{m}, \\ &\operatorname{rot} \vec{v}^{\alpha} + ih[\vec{z}^{0} \times \vec{v}^{\alpha}] + ik\epsilon \vec{u}^{\alpha} = 4\pi c^{-1} \delta_{\alpha e} \vec{j}^{e}, \quad \vec{r}, \vec{r}_0 \in \mathbb{R}^2 \setminus \bar{W}, \end{aligned} \tag{6}$$

$$[\vec{u}^{\alpha} \times \vec{n}]\big|_{\partial M \cup \partial M'} = 0; \quad [[\vec{u}^{\alpha} \times \vec{n}]]\big|_{\partial D} = 0; \quad [[\vec{v}^{\alpha} \times \vec{n}]]\big|_{\partial D} = 0, \tag{7}$$

$$\int_S (\operatorname{Re} \epsilon |\vec{u}^{\alpha}|^2 + |\vec{v}^{\alpha}|^2) ds < \infty, \quad S \subset \mathbb{R}^2, \tag{8}$$

where $\vec{j}^{\alpha}(\vec{r}, \vec{r}_0) = \vec{I}^{\alpha} \delta(\vec{r} - \vec{r}_0)$, $\alpha = e, m$.

Unlike the initial problem (2)-(4), problem (6)-(8) does not contain boundaries receding to infinity, i.e., it is a classical diffraction problem. For this reason a unique solution of this problem can be obtained by applying the radiation principle in the form of Sommerfeld's condition or the condition that the solution should decay exponentially as a function of the ratio of the parameters k and h. Combining both conditions we require that for $r > r_* = \max(a, r_0)$ the solution should be representable in the form of a converging series (see [2, 5]):

$$\vec{g}^{\alpha}(\vec{r}, h) = \sum_{n=-\infty}^{\infty} \{\vec{a}_n^{\alpha}, \vec{b}_n^{\alpha}\} H_n^{(1)}(\kappa r) e^{in\varphi}, \quad \alpha = e, m, \tag{9}$$

where $\kappa^2 = k^2 - h^2 \neq 0$; $\operatorname{Im} k = 0$; $\operatorname{Im} h = 0$; $\operatorname{Re} \kappa \geqslant 0$, $\operatorname{Im} \kappa \geqslant 0$.

The form of condition (9) is explained by the fact that the fundamental solution of (6) $\vec{g}_0^{\alpha}$ with no OW and satisfying the radiation principle can be expressed in terms of Green's function of the two-dimensional Helmholtz equation

$$g_0(\vec{r}, \vec{r}_0; k, h) = i/4 H_0^{(1)}(\kappa |\vec{r} - \vec{r}_0|). \tag{10}$$

Thus the starting three-dimensional problem has been reduced to a simpler two-dimensional problem, which can be formulated in closed form, ensuring that the solution is unique, and subsequent evaluation of integrals of the type (5). This approach has been used many times to study different particular OW [10-17], though here the radiation condition was not stated explicitly.

2. FOURIER TRANSFORMS OF GREEN'S FUNCTIONS AND THEIR ANALYTICAL CONTINUATION IN h. PROBLEMS REGARDING THE SPECTRUM OF GENERALIZED CHARACTERISTIC WAVES

In spite of the fact that problem (6)-(8) was initially formulated for real h, the problem of evaluating integrals of the type (5) unavoidably leads to studying the functions $\vec{g}^{e,m}(h)$ over the entire region of their analytical continuation in h. For example, this is obvious if (5) is evaluated by the methods of contour integration for $|z - z_0| \to \infty$.

It is clear that the region of analyticity of $\vec{g}^{e,m}(h)$ cannot be wider than the region of analytic continuation in h of the fundamental solution (10), which is identical with the infinite-sheet Riemann surface $\mathcal{L}$ of the function $\operatorname{Ln} \kappa(h) = \frac{1}{2}\operatorname{Ln}(k+h)(k-h)$.

We shall describe the surface $\mathcal{L}$. We separate the principle logarithmic branch $\mathcal{L}_0$ of the function $\operatorname{Ln} \kappa$, choosing $-\pi/2 < \arg \kappa < 3\pi/2$, and we associate with it two copies of the sheets $\mathcal{L}_0^{1,2}$

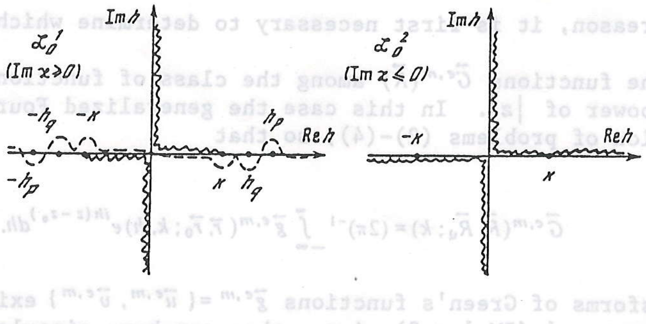

Fig. 2

of the complex h plane in view of the double-valuedness of the function $\kappa(h)$. The sheets $\mathcal{L}_0^1, \mathcal{L}_0^2$ are shown in Fig. 2, where the wavy line shows the cuts and the broken line shows the contour of integration in the Fourier integral. All other pairs of sheets $\mathcal{L}_{n\neq 0}^{1,2}$ differ from $\mathcal{L}_0^{1,2}$ in the $-\pi/2 + 2\pi n < \arg\kappa < 3\pi/2 + 2\pi n$. We draw cuts between $\mathcal{L}_0^1$ and $\mathcal{L}_0^2$ along the lines on which $\operatorname{Im}\kappa = 0$, and on the entire sheet $\mathcal{L}_0^1$ $\operatorname{Im}\kappa \geqslant 0$, while on the sheet $\mathcal{L}_0^2$ $\operatorname{Im}\kappa \leqslant 0$. If $h \in \mathcal{L}_0^{1,2}$, then the formula

$$H_m^{(1)}(\kappa r e^{i2\pi n}) \sim (2/i\pi\kappa r)^{1/2} e^{-i\pi m/2}[(1-2n)e^{i\kappa r} - 2in(-1)^m e^{-i\kappa r}]$$
$$\text{as } |\kappa r| \to \infty \tag{11}$$

holds, which shows that on the sheet $\mathcal{L}_0^1$ the condition (9) describes functions that decay exponentially as $|\kappa r| \to \infty$, while on the sheet $\mathcal{L}_0^2, \mathcal{L}_n^{1,2}$ they are exponentially increasing. For this reason the sheet $\mathcal{L}_0$ is called the "physical" sheet and the contours of integration in (5) are drawn along its real axis, while the other sheets $\mathcal{L}_0^2, \mathcal{L}_n^{1,2}$ are "unphysical" and reflect the, at first glance, anomalous behavior of $\vec{g}^{e,m}(h)$ on them.

We denote by $\mathcal{R}$ the real axis of the "physical" sheet $\mathcal{L}_0^1$. The following theorem is true.

Theorem 1. The Fourier transforms of Green's functions—the solutions of problem (6)-(9)—exist and are unique for $h \in \mathcal{R}, |h| < k$, and can be analytically continued to all sheets of the surface $\mathcal{L}$, with the exception of not more than a discrete set of poles, with a single point of accumulation at infinity. The residues of the complex poles of $\vec{g}^{e,m}(h)$ are the nontrivial solutions of the generalized spectral problem—a homogeneous problem of the type (6)-(9)—for all h from $\mathcal{L}$.

The proof of this theorem is based on the possibility of regularizing problem (6)-(9), i.e. reducing it to the equivalent operator equation

$$[I + T(h)]\,\vec{g}^{\alpha}(\vec{r}) = \vec{g}_0^{\alpha}(\vec{r}); \quad \vec{r} \in D; \quad T = \begin{bmatrix} I + T_1 & 0 \\ T_2 & I \end{bmatrix}, \tag{12}$$

where I is the identity operator and $T(h)$ is an integral operator that depends on h and is compact in the vector Hilbert space $L_2^6(D)$. The regularization is performed by the methods of the theory of generalized ~~three-dimensional~~ volume potentials, so that the components $T(h)$ and $\vec{g}_0^{\alpha}$ are expressed in terms of scalar Green's functions $g_0^{D,N}(h)$ of the Dirichlet and Neumann problems for the two-dimensional Helmholtz equation with the boundary conditions given on ∂M and $\partial M'$ with $\varepsilon \equiv 1$. The latter, in turn, based on [5, 18, 19], exist as meromorphic functions of h on the surface $\mathcal{L}$. For such operator equations Fredholm's theorem is true [20], whence follow all assertions of Theorem 1. In principle, Eq. (12) enables us to find the functions $\vec{g}^{e,m}(h)$ off their poles with any fixed accuracy.

Thus the surface $\mathcal{L}$ off the branch points $h = \pm k$ consists of two sets: the resolvent ρ_h, forming the region of analyticity of $\vec{g}^{e,m}(h)$, and the spectral σ_h, identical to the set of poles of $\vec{g}^{e,m}(h)$.

The problem of finding the poles of $\vec{g}^{e,m}(h)$ can be formulated and studied completely independently, outside of the problem of finding Green's functions. This spectral problem has the form

$$\begin{aligned} &\operatorname{rot}\vec{u} + ih[\vec{z}^{\,0} \times \vec{u}] - ik\vec{v} = 0, \\ &\operatorname{rot}\vec{v} + ih[\vec{z}^{\,0} \times \vec{v}] + ik\varepsilon\vec{u} = 0, \quad r \in \mathbb{R}^2 \setminus \overline{W}, \end{aligned} \tag{13}$$

$$[\vec{u} \times \vec{n}]\big|_{\partial M \cup \partial M'} = 0; \quad [[\vec{u} \times \vec{n}]]\big|_{\partial D} = 0; \quad [[\vec{v} \times \vec{n}]]\big|_{\partial D} = 0, \tag{14}$$

$$\int_S (\operatorname{Re}\varepsilon u^2 + v^2)\,ds < \infty, \quad S \subset \mathbb{R}^2, \tag{15}$$

$$\{\vec{u}, \vec{v}\} = \sum_{n=-\infty}^{\infty} \{\vec{a}_n, \vec{b}_n\}\, H_n^{(1)}(\kappa r) e^{in\varphi}, \quad r > a, \quad h \in \mathcal{L} \setminus \{\pm k\}. \tag{16}$$

The nontrivial solutions of this problem for $h \in \sigma_h$ formally determine the generalized characteristic waves of the OW (see also [5, 21, 22]):

$$\vec{W}(\vec{R}, t) = \{\vec{E}(\vec{R}, t), \vec{H}(\vec{R}, t)\} = \{\vec{u}(\vec{r}), \vec{v}(\vec{r})\} e^{ihz - ikct} = \vec{w}(\vec{r}) e^{ihz - ikct}.$$

We shall adopt the following terminology, depending on the sheet of the surface $\mathcal{L}$ in which the point of the spectrum lies. We shall call points in $\mathcal{R}$ the spectrum of characteristic waves (eigenwaves) of the OW. According to (11), they are a surface character. The points outside $\mathcal{R}$ but on one of the sheets $\mathcal{L}_0^1, \mathcal{L}_0^2$ will comprise the spectrum of quasi-characteristic (quasi-eigenwaves) waves. These waves are surface waves on the "physical" sheet $\mathcal{L}_0^1$ and outgoing waves on the "unphysical" sheet $\mathcal{L}_0^2$. We shall refer the points of the spectrum lying on the remaining sheets $\mathcal{L}_{n\neq 0}^{1,2}$ to the set of pseudo-characteristic (preudo-eigenwaves) waves. According to (11), as $r \to \infty$ their fields will behave as the sum of incoming and outgoing waves, and in this sense they describe some diffraction problem. It may be useful to take these points of the spectrum into account when calculating the fields in the "near zone" of the source.

For the simplest OWs [10-14] the method of separation of variables reduces the problem of finding the spectrum to a dispersion equation with complex h. For example, for a circular, ideally conducting cylinder, σ_h consists of the zeros of $H_m^{(1)}(\kappa a)$ and $H_m^{(1)\prime}(\kappa a)$ and for any m contains not more than a finite number of quasi-characteristic (outgoing) waves and an infinite number of pseudocharacteristic waves. In more complicated cases, when the variables cannot be separated, it is necessary to study the eigenvalues of homogeneous operator equations of the type (12) for $h \in \mathcal{L}$ by the methods of the theory of analytical operator functions (see [22, 23]).

3. A UNIQUENESS THEOREM FOR FOURIER TRANSFORMS OF GREEN'S FUNCTIONS CONTINUED IN h

If some region Ω of the surface $\mathcal{L}$ does not contain points of the spectrum σ_h, then Ω belongs to the resolvent set ρ_h, so that for $h \in \Omega$ the functions $\vec{g}^{\,e,m}(h)$ are unique, if they exist.

Any generalized characteristic wave, being a solution of Maxwell's equations, must satisfy Poynting's theorem

$$\oint_S [\vec{E} \times \vec{H}^*]\,\vec{n}_s\,ds = i\int_V (k|\vec{H}|^2 - \varepsilon^* k^* |\vec{E}|^2)\,dv, \tag{17}$$

where V is an arbitrary volume, bounded by the surface S with the normal $\vec{n}_s$.

Let V be a circular cylinder with radius $r_1 > a$, truncated by the planes $z = 0$, z_1. Let r_1 be large enough so that the expression (11) can be employed when integrating over S. We shall assume that $h \in \mathcal{L}_0^{1,2}$, i.e., we shall study only the characteristic and quasi-characteristic waves of OW. We denote by

$$P_z = c/8\pi \int_{S_1} [\vec{E} \times \vec{H}^*]\,\vec{z}^{\,0}\,ds \tag{18}$$

the complex energy flux through the circle S_1 with radius r_1 in the section of the OW. Integrating in (17) we arrive at the fact that for any characteristic or quasi-characteristic wave the following relations must hold:

$$-\mathrm{Re}\,P_z\,\mathrm{Im}\,h + c(4\pi|\kappa|^3)^{-1}(\mathrm{Re}\,k\,\mathrm{Re}\,\kappa + \mathrm{Im}\,k\,\mathrm{Im}\,\kappa)e^{-2\,\mathrm{Im}\,\kappa r_1} \times$$
$$\times \sum_{n=-\infty}^{\infty} (|a_n|^2 + |b_n|^2) = -c(16\pi)^{-1} \int_{S_1} [\mathrm{Im}\,kH^2 +$$
$$+ (\mathrm{Re}\,k\,\mathrm{Im}\,\epsilon + \mathrm{Im}\,k\,\mathrm{Re}\,\epsilon)E^2]ds, \tag{19}$$
$$-\mathrm{Im}\,P_z\,\mathrm{Im}\,h + c(4\pi|\kappa|^3)^{-1}(\mathrm{Re}\,k\,\mathrm{Im}\,\kappa - \mathrm{Im}\,k\,\mathrm{Re}\,\kappa)e^{-2\,\mathrm{Im}\,\kappa r_1} \times$$
$$\times \sum_{n=-\infty} (|a_n|^2 - |b_n|^2) = -c(16\pi)^{-1} \int_{S_1} [(\mathrm{Re}\,k\,\mathrm{Re}\,\epsilon - \mathrm{Im}\,k\,\mathrm{Im}\,\epsilon)E^2 -$$
$$- \mathrm{Re}\,kH^2]ds. \tag{20}$$

Setting $\mathrm{Im}\,h = 0$ and analyzing the left- and right-hand sides of (19) and (20) for sign-definiteness, we arrive at the following theorem.

Theorem 2. The boundary-value problem (6)-(9) has not more than one solution under the following conditions: 1) for all $h \in \mathcal{R}$, if $\mathrm{Im}\,k > 0$; 2) for $h \neq \pm k$, $h \in \mathcal{R}$, if $\mathrm{Im}\,k = 0$ and either $\mathrm{Im}\,\varepsilon = 0$ or $\varepsilon < 1$; 3) for all $h \in \mathcal{R}$, such that $|h| < k$ or $|h| > k\varepsilon^{1/2}_{\max}$, if $\mathrm{Im}\,k = 0$, $\mathrm{Im}\,\epsilon = 0$, $\epsilon > 1$.

Thus the resolvent set $\rho_h \supset \Omega$ is not empty and contains at least part of the $\mathcal{R}$ axis.

Corollary 2.1. The spectrum of characteristic waves can lie only on the segment $k < |h| < k\varepsilon^{1/2}_{\max}$ of the $\mathcal{R}$ axis with $\mathrm{Im}\,k = 0$ and $\mathrm{Im}\,\varepsilon = 0$ and is thus finite.

Corollary 2.2. The principle of limiting absorption separates a unique solution of problem (6)-(9) only if $h \in \mathcal{R}$ does not belong to the spectrum σ_h, in particular, if either $\mathrm{Im}\,\varepsilon > 0$ or $\varepsilon \leqslant 1$ or $\mathrm{Im}\,\varepsilon = 0$ for $\varepsilon > 1$, but $|h| < k$ or $|h| > k\varepsilon^{1/2}_{\max}$.

Theorem 2 still does not guarantee the existence of characteristic waves of the OW, i.e., real points of the spectrum on $\mathcal{R}$; it merely shows that they can be found. Katsenelenbaum [24] proved that the spectrum of characteristic waves of any dielectric OW contains at least two basic quasi-waves of the the T type, such that for them $h(k) \to k$ as $k \to 0$. The following theorem is true.

Theorem 3. The spectrum of characteristic waves of any metal-dielectric OW of the class studied contains at least $N + 2$ fundamental quasi-waves of the T type, where N is the order of connectedness of the ideally conducting elements of the OW.

The proof of this theorem rests on the application of Ruchet's operator theorem [25] to the analysis of the eigenvalues of (12) as $k \to 0$, $\varepsilon \to 1$. Here it is necessary to take into account the fact that if $\varepsilon \equiv 1$, then the functions $\vec{g}^{e,m}(h)$ have poles of order N at the branch points $h = \pm k$, corresponding to waves of the T type of an N-conducting line located in free space.

4. RADIATION CONDITION AND PRINCIPLE OF LIMITING ABSORPTION FOR OPEN WAVEGUIDES

Thus we have established that the functions $\vec{g}^{e,m}(h)$ exist and are analytic for all h, different from the branch points $\pm k$ and the points of the spectrum σ_h at which the functions have poles. The spectrum does not have finite accumulation points on $\mathcal{L}$. In addition, the fact that the operator $[I + T(h)]^{-1}$ is bounded for $h \notin \sigma_h$ implies that $\|\vec{u}^\alpha\|_{L_2^3(D)} < \mathrm{const}\,\|\vec{u}_0^\alpha\|_{L_2^3(D)}$, which on the "physical" sheet $\mathcal{L}_0^1$ leads to the inequality $|\vec{g}^{e,m}(h)| < \mathrm{const}\,e^{-\mathrm{Im}\,\kappa a}$. These properties enable us to use the saddle-point method to evaluate the integral (5) as $R \to \infty$. In so doing the direction in which the poles and branch points, located on the real $\mathcal{R}$ axis, are circumscribed remains arbitrary: the integral (5) makes sense for any direction. Moreover, because of the equivalence of directions along Oz the singularities that are symmetrical about zero should be circumscribed in opposite directions.

Let the number of poles on each of the semiaxes of $\mathcal{R}$ equal Q. We introduce the vector $\vec{\gamma} = \{\gamma_q\}_{q=0}^{Q}$ where $\gamma_q = \pm 1$, and in addition $\gamma_q = 1$ corresponds to circumscribing the singularity from below while $\gamma_q = -1$ corresponds to circumscribing it from above. We also note that all poles on $\mathcal{R}$ are simple and are different from the branch points $\pm k$. Then applying the standard procedure of the saddle-point method [9-14] we find that as $R = (r^2 + z^2)^{1/2} \to \infty$

$$\vec{G}^\alpha(\vec{R}) \sim \left\{\begin{array}{ll} \vec{\Psi}^\alpha(\varphi,\theta)e^{ik\gamma_0 R}(kR)^{-1}, & r > r_1 \\ o(1), & r < r_1 \end{array}\right\} + \sum_{q=1}^{Q} \beta_q^\alpha \vec{w}_q(\vec{r}, h_q)\Gamma_q(z), \tag{21}$$

where $r_1 = \max(a, r_0)$; $\Gamma_q(z) = e^{i\gamma_q h_q|z-z_0|}$, β_q^α are numerical coefficients, $\alpha = e, m$; $\vec{w}_q(\vec{r}, h_q)$ are the characterstic waves of the OW, i.e., the eigenfunctions of the spectral problem (13)-(16) corresponding to the points $\sigma_h \cap \mathcal{R}$. The contribution of other points of the spectrum to the asymptotic behavior of $\vec{G}^\alpha(\vec{R})$ as $R \to \infty$ is exponentially small, though near the source it can be significant.

It now remains to determine how the arbitrariness of choosing γ_q can be resolved so that (21) can be regarded as a condition ensuring that the solution is unique. For this it is sufficient to show the values of γ_q for which the homogeneous problem (2)-(4) with this condition as $R \to \infty$ has only a trivial solution.

Assume the opposite, i.e., a nontrivial solution exists, and apply Poynting's theorem. Consider the real part of (17) for the region V_* bounded by the surface S_* (Fig. 3). Its right-hand side with $\operatorname{Im} k = 0$, $\operatorname{Im} \varepsilon = 0$ equals zero. On the left-hand side we pass to the limit as $S_* \to \infty$ in such a manner that $R_* \to \infty$, $r_* \to \infty$, $r_*/R_* \to 0$. Then, using (21) and also the property that the characteristic waves in the cross-section of the OW are orthogonal [9], we obtain

$$\frac{c}{8\pi} \lim_{S_* \to \infty} \oint_{S_*} [\vec{E} \times \vec{H}^*] \vec{n}_s ds = \frac{c\gamma_0}{8\pi k^2} \int_0^{\pi} \int_0^{2\pi} [|\Psi_{Ez}|^2 + |\Psi_{Hz}|^2] \sin\theta d\theta d\varphi + 2 \sum_{q=1}^{Q} \gamma_q |\beta_q|^2 \operatorname{Re} P_{zq} = 0. \qquad (22)$$

The left-hand side of expression (22) is sign-definite only if either $\gamma_0 = 1$, $\gamma_q = \operatorname{sign} \operatorname{Re} P_{zq}$ or $\gamma_0 = -1$, $\gamma_q = -\operatorname{sign} \operatorname{Re} P_{zq}$, which contradicts the initial assumption. The first method for choosing γ_k gives a solution for which energy is carried off to infinity while the second method gives a solution for which energy arrives from infinity, i.e., it does not conform to the radiation principle. This proves the following theorem.

Theorem 4. Let $\operatorname{Im} k = 0$, $\operatorname{Im} \varepsilon = 0$ and all $2Q$ points of the spectrum σ_h, lying on the $\mathcal{R}$ axis, be simple and different from the branch points $\pm k$. Then there exists a unique solution of the problem (2)-(4) $\vec{G}^\alpha(R)$, conforming to the radiation principle, understood as the requirement that there be no waves delivering energy, and satisfying as $R \to \infty$ condition (21), where $\gamma_0 = 1$, $\gamma_q = \operatorname{sign} \operatorname{Re} P_{zq}$ $(q = 1, \ldots, Q)$.

Thus in the presence of an OW the field sufficiently far away from sources should have the form of a sum of a spherical wave, satisfying the Sommerfeld condition, and a finite number of characteristic waves of the OW. For each of the latter a partial radiation condition can be formulated for $|z| \to \infty$:

$$\left(\frac{d}{d|z|} - i\gamma_q h_q \right) \Gamma_q(z) = 0, \quad q = 1, \ldots, Q.$$

This condition is analogous to Sveshnikov's condition for closed waveguides [3, 4], but it differs from the latter in that the quantities γ_q are present. Thus the requirement that

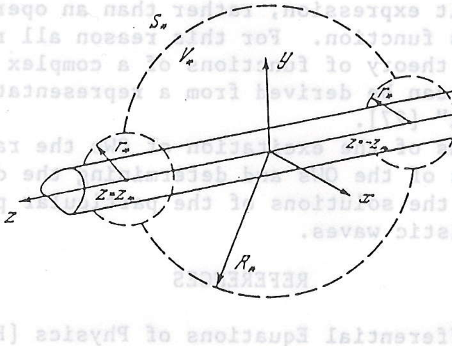

Fig. 3

there be no waves delivering energy is more general than the requirement that there be no arriving waves (see Mandel'shtam's lectures [26]).

The spectrum may not contain any real points (see Theorem 2). The following corollary is obvious.

Corollary 4.1. Let $\mathrm{Im}\, k = 0$ and either $\mathrm{Im}\, \varepsilon > 0$ or $\mathrm{Im}\, \varepsilon = 0$, $\varepsilon \leqslant 1$. Then a unique solution of problem (2)-(4) that satisfies the radiation principle can be separated with the help of Sommerfeld's radiation condition.

Thus Theorem 4 guarantees that the solution of the starting problem with $\mathrm{Im}\, k = 0$ is unique. For $\mathrm{Im}\, k > 0$ the solution is also unique in the class of functions that decay at infinity. Further, since it can be shown that the points of the spectrum depend in a piecewise-analytical fashion on k, by studying the limit in (19) the concept of a group velocity can be introduced for the characteristic waves of the OW:

$$v_q \equiv (dh_q/dk)^{-1} = 16\pi \mathrm{Re}\, P_{zq} \left\{ c \int_{\mathbb{R}^1} (\mathrm{Re}\, \epsilon u_q^2 + v_q^2)\, ds \right\}^{-1}. \tag{23}$$

Expression (23) shows that as $\mathrm{Im}k \to +0$, the poles approaching $\mathcal{R}$ deform the integration contour in (5) in the precise manner required by Theorem 4.

Corollary 4.2. The principle of limiting absorption separates a unique solution of problem (2)-(4), whose limit as $\mathrm{Im}k \to +0$ exists and its asymptotic behavior as $R \to \infty$ is given by (21), where $\gamma_0 = 1$, $\gamma_q = \mathrm{sign}\, \mathrm{Re}\, P_{zq}$ $(q = 1, \ldots, Q)$.

Further, it can be shown by direct substitution that for any solutions of Eqs. (2), $\{\vec{E}_1, \vec{H}_1\}$ and $\{\vec{E}_2, \vec{H}_2\}$, satisfying condition (21), the relation

$$\lim_{S_* \to \infty} \oint_{S_*} \{[\vec{E}_1 \times \vec{H}_2] - [\vec{E}_2 \times \vec{H}_1]\} \vec{n}_s ds = 0 \tag{24}$$

holds.

Relation (24) together with the vector Green's formula enable us to construct a solution of problem (2)-(4) with arbitrary finite functions $\vec{J}^e, \vec{J}^m$ on the right-hand side of the form of a convolution with Green's tensor (1):

$$\begin{bmatrix} \vec{E}(\vec{R}) \\ \vec{H}(\vec{R}) \end{bmatrix} = \int_{V_0} \begin{bmatrix} \hat{G}_{ee} \hat{G}_{em} \\ \hat{G}_{me} \hat{G}_{mm} \end{bmatrix} \begin{bmatrix} \vec{J}^e(\vec{R}_0) \\ \vec{J}_m(\vec{R}_0) \end{bmatrix} dv_0, \quad V_0 \subset \mathbb{R}^3 \setminus (\bar{W} \times z). \tag{25}$$

Then, by virtue of linearity, the total field $\{\vec{E}, \vec{H}\}$ also satisfies the radiation condition (21), consistent with the radiation principle.

In conclusion we shall present without proof an expression, analogous to (21), for two-dimensional problems of the excitation of plane-layered OW. For $\mathrm{Im}k = 0$, $\mathrm{Im}\epsilon = 0$ and $r = (z^2 + y^2)^{1/2} \to \infty$

$$G^\alpha(\vec{r}) \sim \begin{cases} \Psi^\alpha(\varphi) e^{ikr} (kr)^{-1/2}, & y > y_1 \\ o(1), & y < y_1 \end{cases} + \sum_{q=1}^{Q} \beta_q^\alpha u_q(y, h_q) e^{i\gamma_q h_q |z - z_0|}. \tag{26}$$

In such problems an explicit expression, rather than an operator equation, is obtained for the Fourier transform of Green's function. For this reason all results necessary to prove (26) follow from the theorems of the theory of functions of a complex variable. We note that condition (26) for plane-layered OWs can be derived from a representation of the fields as a sum of "discrete and continuous spectra" [27].

Thus in formulating problems of the excitation of OWs the radiation condition requires finding all characteristic waves of the OWs and determining the direction in which these waves transport energy. We note that the solutions of the particular problems [9-16] satisfy (21) if there are no backward characteristic waves.

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