Dual Integral Equations and Analytical Regularization Technique in the Study of Purcell Effect for a Thin Dielectric Disk

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Abstract. We consider spontaneous radiation of atomic or molecular dipole in the presence of a thin dielectric microdisk as a boundary-value problem for 3D Maxwell's equations, local energy finiteness, and a radiation condition at infinity. The problem is reduced first to dual integral equations (IEs) and then to regularized IEs and the latter are solved with a meshless Nystrom algorithm. Results show that the radiative and non-radiative decay rates display resonance maxima associated with the disk natural frequencies explained through the effective-refractive-index approximation.

Keywords: dielectric disk, generalized boundary conditions, dual integral equations, spontaneous emission
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INTRODUCTION

Enhancement or, generally speaking, modification of the spontaneous emission of an atomic or molecular dipole due to resonant environment is usually called the Purcell effect [1-5]. Such enhancement is important for better understanding of limiting performance of microlasers and in cavity quantum electrodynamics. In the past, this effect was usually explained via so-called "Purcell factor," proportional to the ratio of the mode quality factor Q to the mode volume V. However, it has been recently convincingly argued that this factor, originally derived for closed cavities with imperfectly conducting walls, cannot be used in the case of open ones. To study this effect from the viewpoint of Maxwell equations, one has to calculate the radiative and non-radiative decay rates in the presence of an open resonator [3-5]. This can be done using various computational methods, however the use of the full-wave and convergent computational techniques is mandatory if high-Q modes are present.

PROBLEM STATEMENT

We consider the scattering of electromagnetic field emitted by an elementary electric or magnetic dipole (EED or EMD) located at the height h above a thin magneto-dielectric disk of radius a and thickness τ (Fig. 1, where the on-axis horizontal electrical dipole is shown as a source of the incident field). We denote total electromagnetic field as a sum, $E = E_{in} + E_{sc}$, $H = H_{in} + H_{sc}$, where the incident field

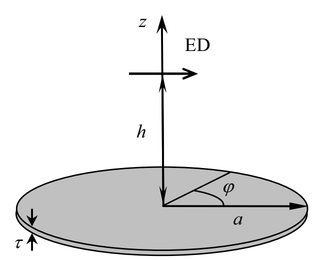

FIGURE 1. Geometry of a dipole above a dielectric microdisk.

corresponds to the EED or EMD in the free space and introduce dimensionless cylindrical coordinates $(\rho = r/a, \varphi, \zeta = z/a)$ with the origin in the disk center. To be uniquely determined, the scattered field has to satisfy the homogeneous Maxwell equations out of the sources and the disk, continuity of tangential components at the disk boundary, 3-D radiation condition at infinity, and the condition of local integrability of power.

This boundary-value problem can be reduced to the volume and boundary IEs and attacked numerically. Still this is a challenge for researcher that is proved by the absence of published analysis of the 3-D modes of a disk of realistic size. However, inspection of photonics applications shows that frequently dielectric or semiconductor disks are thin flat cavities produced by the etching and epitaxy techniques. If the disk thickness is smaller than the emission wavelength, it can be shrunk to the median section where the following two-side generalized boundary conditions (GBC) can be imposed [6]:

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$$(E_{tg}^{+} + E_{tg}^{-}) = 2Z_0 R \cdot \vec{n} \times (H_{tg}^{+} - H_{tg}^{-}), \qquad (1)$$
$$Z_0(H_{tg}^{+} + H_{tg}^{-}) = -2Q \cdot \vec{n} \times (E_{tg}^{+} - E_{tg}^{-}).$$

Here, Z_0 is the free-space impedance, and Z is the relative impedance of the disk material, $k = \omega / c$ is the wavenumber, R and Q are so-called electric and magnetic resistivities given by $R = iZ/2\cot(\sqrt{\varepsilon_r \mu_r} k\tau / 2)$, $Q = R / Z^2$ in the case of $k\tau \ll 1$ and $|\varepsilon_r \mu_r| \gg 1$, ε_r and μ_r are the relative permittivity and permeability, respectively. On the rest part of the disk plane the components of the total field must be continuous.

Further we introduce the normal to the disk field components in terms of Fourier-Bessel transform,

$$\begin{pmatrix} E_{sc,z}^{\mathrm{sgn}(\zeta)} \\ Z_0 H_{sc,z}^{\mathrm{sgn}(\zeta)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta|} J_m(\kappa\rho) \begin{pmatrix} \kappa e_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \\ \kappa h_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \end{pmatrix} d\kappa \quad (2)$$

and present the tangential to the disk field components via the vector Hankel transform (3) and (4), where

$$\mathbf{H}_m(\kappa\rho) = \begin{pmatrix} J'_{|m|}(\kappa\rho) & mJ_{|m|}(\kappa\rho)/(\kappa\rho) \\ mJ_{|m|}(\kappa\rho)/(\kappa\rho) & J'_{|m|}(\kappa\rho) \end{pmatrix}$$

Similar expressions can be written for the incident field. Note that thus presented fields satisfy the radiation condition of Silver-Muller automatically.

Substituting tangential to the disk field components to GBC, we obtain a set of coupled dual IEs (5) and (6), where

$$u_m^{sc,\pm}(\kappa) = \left(e_{m,z}^{sc,+}(\kappa) \pm e_{m,z}^{sc,-}(\kappa)\right)/2,$$
$$v_m^{sc,\pm}(\kappa) = \left(h_{m,z}^{sc,+}(\kappa) \pm h_{m,z}^{sc,-}(\kappa)\right)/2$$

are four unknown functions of the set of coupled dual IEs and $u_m^{in,\pm}(\kappa)$, $v_m^{in,\pm}(\kappa)$ are determined by the incident field functions.

To find the solutions of obtained IEs (5) and (6) we firstly reduced them to the Fredholm second kind IEs (FIE-2) following the scheme outlined in [7]:

- Integrate each of the dual IEs with respect to radius variable and obtain four other coupled IEs,
- Split the integral operators of each coupled IEs into regular and singular parts and invert the latter parts using the method of analytical regularization [7].

Favorable features of FIE-2 guarantee the uniqueness and existence of their solutions and convergence of any numerical algorithm based on a reasonable discretization scheme. In our case we do the following:

- Introduce the truncation number $N \geq ka + 1$ and truncate of the interval integration to $(0,N)$.
- Apply the Nystrom method with the Gauss-type higher-order quadratures to discretize FIE-2 on the $(0,N)$ interval, and find the unknowns at the grid points by inverting the matrix analog of IEs.
- Find the unknown functions on the $(0,\infty)$ interval by substitution of the set of found values into FIE-2.

Note that such developed meshless numerical technique leads to the accurate well convergent and fast algorithm. We have found that only $5ka\sqrt{\varepsilon}/\pi$ unknowns are needed to compute the near (far) fields with a uniform relative accuracy of not less than 4 (5) correct digits. This is supported by the plots in [8], where the dependences of the relative computational errors on the order of discretization scheme N have been presented. As one can see, an exponential convergence takes place.

$$\begin{pmatrix} E_{sc,r}^{\mathrm{sgn}(\zeta)} \\ -iE_{sc,\phi}^{\mathrm{sgn}(\zeta)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta|} \mathbf{H}_m(\kappa\rho) \begin{pmatrix} \mathrm{sgn}(\zeta)\, i\gamma(\kappa) e_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \\ -ka\, h_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \end{pmatrix} d\kappa \quad (3)$$

$$\begin{pmatrix} Z_0 H_{sc,r}^{\mathrm{sgn}(\zeta)} \\ -iZ_0 H_{sc,\phi}^{\mathrm{sgn}(\zeta)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta|} \mathbf{H}_m(\kappa\rho) \begin{pmatrix} \mathrm{sgn}(\zeta)\, i\gamma(\kappa) h_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \\ ka\, e_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \end{pmatrix} d\kappa \quad (4)$$

$$\begin{cases} \displaystyle\int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} \gamma(\kappa)\left(u_m^{sc,-}(\kappa) + u_m^{in,-}(\kappa)\right) + 2R\, ka\, u_m^{sc,-}(\kappa) \\ ika\left(v_m^{sc,+}(\kappa) + v_m^{in,+}(\kappa)\right) + 2R\, i\gamma(\kappa) v_m^{sc,+}(\kappa) \end{pmatrix} d\kappa = \bar{0} & (\rho < 1) \\ \displaystyle\int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} ika\, u_m^{sc,-}(\kappa) \\ -\gamma(\kappa) v_m^{sc,+}(\kappa) \end{pmatrix} d\kappa = \bar{0} & (\rho > 1) \end{cases} \quad (5)$$

$$\begin{cases} \displaystyle\int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} \gamma(\kappa)\left(v_m^{sc,-}(\kappa) + v_m^{in,-}(\kappa)\right) + 2Q\, ka\, v_m^{sc,-}(\kappa) \\ -\left(ika\left(u_m^{sc,+}(\kappa) + u_m^{in,+}(\kappa)\right) + 2Q\, i\gamma(\kappa) u_m^{sc,+}(\kappa)\right) \end{pmatrix} d\kappa = \bar{0} & (\rho < 1) \\ \displaystyle\int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} ika\, v_m^{sc,-}(\kappa) \\ \gamma(\kappa) u_m^{sc,+}(\kappa) \end{pmatrix} d\kappa = \bar{0} & (\rho > 1) \end{cases} \quad (6)$$

NUMERICAL RESULTS

Some results of computations are shown in this section. Figs. 3 and 4 present, respectively, the dependences of the radiative (P_{rad}) and non-radiative, i.e. absorptive (P_{abs}) decay rates on the normalized frequency ka for the dielectric disks with thickness $\tau = 0.0058a$ and permittivity $\varepsilon_r = 12(1 + i*10^{-5})$ illuminated by the EED and EMD which are located above the disk on distance $h = 0.005a,\ 0.02a$. Decay rates are normalized by the free-space rate P_0. One can see the resonances in both the radiative and absorptive decay rates at certain frequencies. The nature of thin-disk resonances can be explained using the effective index theory. According to this theory, the thin-disk characteristic equation for the modes of arbitrary azimuth order m (i.e. varying as $\cos m\varphi$) is approximately given as $J_m(\alpha_{eff}^H ka) \approx 0$, where J_m is the Bessel function of the order m and $\alpha_{eff}^H < \sqrt{\varepsilon_r}$ is the effective refractive index. In our case the azimuth field dependence corresponds to $m = 1$ and the effective refractive index is found from GBC to be $\alpha_{eff}^H = [1 - 1/(4R^2)]^{1/2}$.

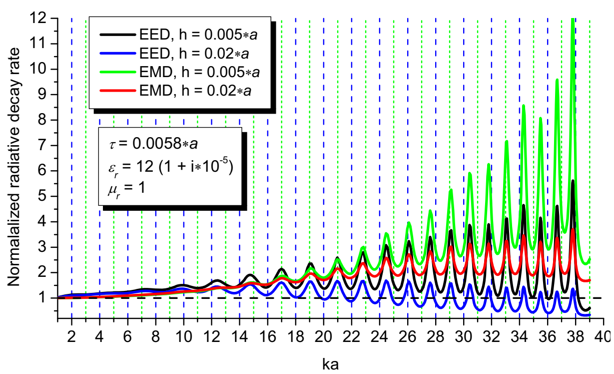

FIGURE 3. Normalized radiative decay rate vs. the dimensionless frequency parameter, *ka***.**

In summary, we have accurately computed the modification of spontaneous emission rates (both radiative and absorptive) of elementary horizontal electric and magnetic dipoles in the presence of a microsize dielectric disk resonator of nanoscale thickness. This has been done over a wide range of the wavelengths to disk-radius ratio values: e.g. if the disk diameter is several microns then the range of associated wavelength covers the whole visible range.

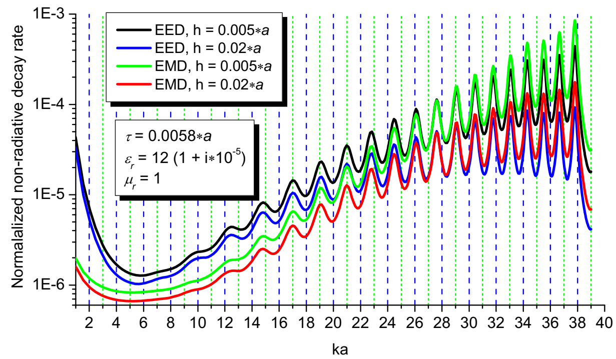

FIGURE 4. Normalized non-radiative decay rate vs. *ka***.**

The resonances found in Figs. 3 and 4 show that the spontaneous emission rates can be enhanced by orders of magnitude depending on the disk thickness and material. They are explained via the effective refractive index model of the disk. These resonances are caused by the standing waves formed due to the reflection of the fundamental guided wave of the dielectric slab by the disk rim. This behaviour is very far from a simplified pattern suggested by the use of Purcell factor as a universal figure-of-merit.

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