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RIGOROUS SUBSTANTIATION OF THE METHOD OF EXACT ABSORBING CONDITIONS IN TIME-DOMAIN ANALYSIS OF OPEN ELECTRODYNAMIC STRUCTURES

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Abstract—Exact absorbing conditions are used in computational electrodynamics of nonsine waves for truncating the domain of computation when replacing the original open initial boundary value problem by a modified problem formulated in a bounded domain. In this paper we prove the equivalency of these two problems.

1. INTRODUCTION

The efficient limitation of the computational domain in open initial boundary value problems (i.e., the problems whose domain of analysis is infinite in one or more directions) is a vital issue in computational electrodynamics as well as in other physical disciplines using mathematical simulation and numerical experiments. Most of the well-known and extensively used heuristic and approximate solutions to this problem are based on the so-called Absorbing Boundary Conditions (ABC) [1–4] and Perfectly Matched Layers (PML) [5–7]. The use of various modifications and improving techniques for these methods yield good results in various specific physical situations. However, it appears that for certain problems associated with the resonant wave scattering, the numerical implementation of these methods may cause unpredictable growth of the computational error for large observation times.

The method utilizing the exact absorbing conditions for the artificial boundaries that truncate an unbounded domain of

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computation [8–11] is outnumbered by the classical approximate approaches. However, the testing data as well as a series of physical and applied results obtained with the help of this method (see, for example [9, 11–14]), shows its evident potential, especially, for obtaining reliable numerical data on space-time and space-frequency electromagnetic field transformations in open waveguide, periodic, and compact resonators.

The essence of the method is as follows. Assume that excitation sources and medium inhomogeneities are located in a bounded region Ω_{int} of the unbounded domain of analysis Ω and take into consideration that by the time $t \leq T < \infty$ the excitation wave $U(g, t)$ has not yet reached the points $g \in \tilde{\Omega}$ of the domain Ω. Then the well-known exact radiation condition

$$U(g,t)|_{g\in\tilde{\Omega},\ t\in[0,T]} = 0 \tag{1}$$

for the outgoing pulsed waves at these points is transferred onto some artificial boundary Γ located in the region, where the intensity of space-time field transformations can be arbitrary in magnitude:

$$M\left[U\left(g,t\right)\right]|_{g\in\Gamma} = 0, \quad t \geq 0 \tag{2}$$

Here $U(g, t)$ $(g \in \Omega,\ t \geq 0)$ is a scalar or vector field function, while M is some integro-differential operator on $\Gamma \times [0, \infty)$. The boundary Γ divides the unbounded domain Ω into two domains, namely, Ω_{int} and Ω_{ext} such that $\Omega = \Omega_{int} \cup \Omega_{ext} \cup \Gamma$. In the first one (bounded), we can formulate the initial boundary value problem with respect to the function $U(g, t)$ with the help of boundary condition (2). This problem will be called the modified problem as distinct from the original initial boundary value problem formulated in the unbounded domain Ω with the radiation condition (1) involved. In the domain Ω_{int}, the desired function $U(g, t)$ can be determined by using standard finite-difference algorithm [15]. In the domain Ω_{ext}, we use the so-called 'transport operators' $Z_{p\in\Gamma\rightarrow g\in\Omega_{ext}}(t)$ [8, 9] to determine the values of the function $U(g, t)$ from its values on the boundary Γ:

$$U\left(g,t\right) = Z_{p\in\Gamma\rightarrow g\in\Omega_{ext}}\left(t\right)\left[U\left(p,\tau\right)\right], \quad 0 \leq \tau \leq t. \tag{3}$$

The analytical forms of the operators M and Z depend on the geometry of the domain Ω_{ext}, and, evidently, on the problem dimensions and the coordinates system. However, in all cases, the derivation of these operators is based on the common sequence of transformations widely used in the theory of hyperbolic equations: (A) the isolation of the regular domain Ω_{ext} where the wave $U(g, t)$ propagates freely moving away from the domain Ω_{int} enveloping all sources and scattering objects; (B) incomplete separation of variables in the original initial boundary value problem for the domain

Ω_{ext} resulting in the problem for the one-dimensional Klein-Gordon equation with respect to the space-time amplitudes of the field $U(g, t)$; (C) integral transformations (image $\leftrightarrow$ original function) in the problems for one-dimensional Klein-Gordon equations; (D) solution of auxiliary boundary value problems for ordinary differential equations with respect to the images of amplitudes of the field $U(g, t)$; (E) inverse integral transformations. As a result, the nonlocal (in space and time) exact absorbing conditions on the artificial boundary Γ are derived. In some cases, these nonlocal conditions can be reduced to the local conditions by replacing certain integral forms with the differential ones and defining an additional initial boundary value problem with respect to some auxiliary function of time and transverse coordinates [9]. The exact absorbing conditions (2) can be then included into a standard finite-difference algorithm with the domain of calculation reduced down to Ω_{int}. However, one can confidently assert that these finite-difference computational schemes are stable and convergent only when the modified problem is uniquely-solvable and equivalent to the original problem [16]. Although the corresponding assertions have been formulated in some papers on the subject (see, for example [8, 9]), they have in no case been proved analytically. In the present paper we are making up this deficiency for the initial boundary value problems describing pulse wave scattering in open axially-symmetrical structures [10]. There is a reason to believe that this proof scheme can be used for the other types of exact absorbing conditions as well.

2. FORMULATION OF THE MODEL INITIAL BOUNDARY VALUE PROBLEM

In Fig. 1, the cross-section of a model for an open axially-symmetrical ($\partial/\partial\phi \equiv 0$) resonant structure is shown, where $\{\rho, \phi, z\}$ are cylindrical and are $\{r, \vartheta, \phi\}$ spherical coordinates. By $\Sigma = \Sigma_\phi \times [0, 2\pi]$ we denote perfectly conducting surfaces obtained by rotating the curve Σ_ϕ about the z-axis; the relative permittivity $\varepsilon(g)$, $g = \{\rho, z\}$ and specific conductivity $\sigma_0(g) = \eta_0^{-1}\sigma(g)$ are smooth enough (in TE_{0n}-case, clarified below) or constant (in TM_{0n}-case) nonnegative functions inside Ω_{int} and take free space values ($\varepsilon = 1$ and $\sigma_0 = 0$) outside; $\eta_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the impedance of free space, ε_0 and μ_0 are the electric and magnetic constants of vacuum.

The two-dimensional initial boundary value problem describing the distribution of pulsed axially-symmetrical TE_{0n}-($E_\rho = E_z = H_\phi \equiv 0$) and TM_{0n}-waves ($H_\rho = H_z = E_\phi \equiv 0$) in the open

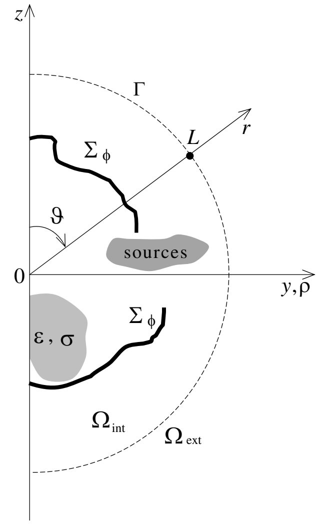


Figure 1. Geometry of the problem in the half-plane $\phi = \pi/2$.

structures of this kind is given by

$$\begin{cases} \left[-\varepsilon(g)\dfrac{\partial^2}{\partial t^2} - \sigma(g)\dfrac{\partial}{\partial t} + \dfrac{\partial^2}{\partial z^2} + \dfrac{\partial}{\partial \rho}\left(\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\rho\right)\right] U(g,t) = F(g,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0, \quad g \in \Omega \\ U(g,t)\big|_{t=0} = \varphi(g), \quad \dfrac{\partial}{\partial t} U(g,t)\Big|_{t=0} = \psi(g), \quad g = \{\rho, z\} \in \bar{\Omega} \\ E_{tg}(p,t)\big|_{p=\{\rho,\phi,z\}\in\Sigma} = 0, \quad t \geq 0 \\ U(0,z,t) = 0, \quad |z| < \infty, \quad t \geq 0, \end{cases} \tag{4}$$

where $\vec{E} = \{E_\rho, E_\phi, E_z\}$ and $\vec{H} = \{H_\rho, H_\phi, H_z\}$ are the electric and magnetic field vectors; $U(g,t) = E_\phi(g,t)$ for TE_{0n}-waves and $U(g,t) = H_\phi(g,t)$ for TM_{0n}-waves [9]. The SI system of units is used. The variable t being the product of the real time by the velocity of light in free space, has the dimensions of length.

The domain of analysis Ω is the part of the half-plane $\phi = \pi/2$ bounded by the contours Σ_ϕ. The regions Ω_{int} and Ω_{ext} (free space) are separated by the virtual boundary $\Gamma = \{g = \{r, \vartheta\} \in \Omega \colon r = L\}$, where $\Omega_{int} = \{g = \{r, \vartheta\} \in \Omega \colon r < L\}$ and $\Omega = \Omega_{int} \cup \Omega_{ext} \cup \Gamma$.

The functions $F(g, t)$, $\varphi(g)$, $\psi(g)$, $\sigma(g)$, and $\varepsilon(g) - 1$ which are finite in the closure $\bar{\Omega}$ of Ω are supposed to satisfy the theorem on the unique solvability of problem (4) in the Sobolev space $\mathbf{W}_2^1(\Omega^T)$, $\Omega^T = \Omega \times (0, T)$, $T < \infty$ (see Statement 1 below and [9, 16]). The 'current' and 'instantaneous' sources given by the functions $F(g, t)$ and $\varphi(g)$, $\psi(g)$ as well as all scattering elements given by the functions $\varepsilon(g)$, $\sigma(g)$ and by the contours Σ_ϕ are located in the region Ω_{int}. In axially-symmetrical problems, at the points $g = \{\rho, z\}$ such that $\rho = 0$, only H_z or E_z fields components are nonzero. Hence it follows that $U(0, z, t) = 0$; $|z| < \infty$, $t \geq 0$ in (4).

Let us assume that $0 < \nu \leq 1/\varepsilon(g) \leq \mu < \infty$ $(g \in \Omega)$ and the functions σ/ε, $\varepsilon'/\varepsilon^2$ are bounded in Ω. Then the following statement [9, 16] is true.

Statement 1. *Let* $F(g,t)/\varepsilon(g) \in \mathbf{L}_{2,1}(\Omega^T)$, $\varphi(g) \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ *(for* TE_{0n}*-waves) or* $\varphi(g) \in \mathbf{W}_2^1(\Omega)$ *(for* TM_{0n}*-waves) and* $\psi(g) \in \mathbf{L}_2(\mathbf{Q})$. *Then problem (4) for all* $t \in [0, T]$ *has a generalized solution from* $\mathbf{W}_2^1(\Omega^T)$, *and the uniqueness theorem is true in this space.*

Here the following notations are used: ε' is the partial derivative of $\varepsilon(g)$ with respect to ρ or z; $\mathbf{L}_n(G)$ is the space of functions $f(g)$, $(g \in G)$ for which the function $|f(g)|^n$ is integrable in G; $\mathbf{W}_m^l(G)$ is the set of all the elements $f(g)$ from $\mathbf{L}_m(G)$ having generalized derivatives up to the order l including, from $\mathbf{L}_m(G)$; $\mathbf{L}_{2,1}(G^T)$ is the space containing all elements $f(g,t) \in \mathbf{L}_1(Q^T)$ with finite norm $\|f\| = \int\limits_0^T (\int\limits_G |f|^2 dg)^{1/2} dt$; $\overset{\circ}{\mathbf{W}}{}_2^1(G)$ is the subspace of space $\mathbf{W}_2^1(G)$, in which the set of compactly supported and infinitely differentiable in G functions is a dense set.

3. EXACT ABSORBING CONDITION FOR AN ARTIFICIAL BOUNDARY AND THE TRANSPORT OPERATOR

In the domain Ω_{ext}, where the outgoing waves $U(g, t)$ propagate freely up to infinity as $t \to \infty$, the 2-D initial boundary value problem (4) can be rewritten in the spherical coordinates:

$$\begin{cases} \left[-\dfrac{\partial^2}{\partial t^2} + \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2} r + \dfrac{1}{r^2}\dfrac{\partial}{\partial \vartheta}\left(\dfrac{1}{\sin\vartheta}\dfrac{\partial}{\partial\vartheta}\sin\vartheta\right)\right] U(g,t) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad t > 0, \quad g \in \Omega_{ext} \\ U(g,t)|_{t=0} = 0, \quad \dfrac{\partial}{\partial t} U(g,t)\,|_{t=0} = 0, \quad g \in \overline{\Omega_{ext}} \\ U(r,0,t) = U(r,\pi,t) = 0, \quad r \geq L, \quad t > 0. \end{cases} \tag{5}$$

By separating the variable ϑ, we can represent the solution $U(g, t)$ as

$$U(r, \vartheta, t) = \sum_{n=1}^{\infty} u_n(r, t)\, \mu_n(\cos\vartheta), \qquad r \geq L, \tag{6}$$

where $\{\mu_n(\cos\vartheta)\}_{n=1,2,\dots}$ is a complete in $\mathbf{L}_2\,(0<\vartheta<\pi)$ orthonormal (with the weight $\sin\vartheta$) system of functions $\mu_n(\cos\vartheta) = \sqrt{(2n+1)/(2n(n+1))}\,P_n^1(\cos\vartheta)$. It is defined by the nontrivial solutions of the homogeneous Sturm-Liouville problem

$$\begin{cases} \left[\dfrac{d^2}{d\vartheta^2} + \mathrm{ctg}\vartheta\dfrac{d}{d\vartheta} - \dfrac{1}{\sin^2\vartheta} + \lambda^2\right]\mu(\cos\vartheta) = 0, \quad 0<\vartheta<\pi \\ \mu(\cos\vartheta)|_{\vartheta=0,\pi} = 0. \end{cases}$$

The space-time amplitudes

$$u_n\,(r,t) = \int\limits_0^\pi U\,(r,\vartheta,t)\,\mu_n\,(\cos\vartheta)\sin\vartheta d\vartheta \tag{7}$$

of the partial components of the spherical wave $U\,(g,\,t)$ are determined by the solutions to the following initial boundary value problems:

$$\begin{cases} \left[-\dfrac{\partial^2}{\partial t^2} + \dfrac{\partial^2}{\partial r^2} - \dfrac{\lambda_n^2}{r^2}\right] r\,u_n\,(r,t) = 0, \quad r\ge L, \quad t>0 \\ u_n\,(r,0) = \left.\dfrac{\partial}{\partial t}u_n\,(r,t)\right|_{t=0} = 0, \qquad r\ge L\ . \end{cases} \tag{8}$$

Here, $P_n^1\,(x)$ are the associated Legendre functions of the first kind and $\lambda_n = \sqrt{n(n+1)}$ are the eigenvalues associated with the functions $\mu_n\,(\cos\vartheta)$ [17].

Denote $w_n(r,t) = ru_n(r,t)$. For $\gamma = n+1/2$, let us apply to (8) the following integral transform

$$\tilde{f}(\omega) = \int\limits_L^\infty f\,(r)\,\sqrt{r}J_\gamma\,(\omega r)\,dr = \frac{1}{\sqrt{\omega}}\int\limits_0^\infty f\,(r)\,\chi\,(r-L)\,\sqrt{r\omega}J_\gamma\,(\omega r)\,dr \tag{9}$$

and pass (see paper [10]) to the following Cauchy problems for the images $\tilde{w}_n\,(\omega,\,t)$ $(\omega\ge 0)$:

$$\begin{cases} \left[\dfrac{\partial^2}{\partial t^2} + \omega^2\right]\tilde{w}_n\,(\omega,t) \\ = w_n\,(L,t)\left[\dfrac{1}{2\sqrt{L}}J_\gamma\,(\omega L) + \omega\sqrt{L}J'_\gamma\,(\omega L)\right] \\ -\sqrt{L}J_\gamma\,(\omega L)\left.\dfrac{\partial w_n\,(r,t)}{\partial r}\right|_{r=L} = g\,(\omega,t)\,, \qquad t>0 \\ \tilde{w}_n\,(\omega,0) = \left.\dfrac{\partial}{\partial t}\tilde{w}_n\,(\omega,t)\right|_{t=0} = 0. \end{cases} \tag{10}$$

Here, $\chi(x) = \begin{cases} 0 \text{ for } & x < 0 \\ 1 \text{ for } & x \geq 0 \end{cases}$ is the Heaviside step-function, J_γ is the Bessel function, and symbol '$'$' denotes derivatives with respect to the whole argument ωL. The generalized statement for problems (10) (the functions $\tilde{w}_n(\omega, t)$ and $g_n(\omega, t)$ are extended with zero on the semiaxis $t \leq 0$) takes the form [18]:

$$\begin{aligned} \left[\frac{\partial^2}{\partial t^2}+\omega^2\right] \tilde{w}_n(\omega, t) &= g(\omega, t)+\delta^{(1)}(t)\tilde{w}_n(\omega, 0)+\delta(t) \left.\frac{\partial}{\partial t}\tilde{w}_n(\omega, t)\right|_{t=0} \\ &= g(\omega, t), \quad -\infty < t < \infty, \end{aligned} \tag{11}$$

where δ and $\delta^{(1)}$ are the δ-Dirac function and its generalized derivative respectively. The convolution of the fundamental solution $G(\omega, t) = \chi(t)\omega^{-1} \sin \omega t$ of the operator $D(\omega) = [\partial^2/\partial t^2 + \omega^2]$ [9, 18] with the right-hand side of Equation (11) allows us to write $\tilde{w}_n(\omega, t)$ in the form

$$\begin{aligned} \tilde{w}_n(\omega, t) &= \frac{1}{2\omega\sqrt{L}} J_\gamma(\omega L) \int_0^t w_n(L, \tau) \sin[\omega(t-\tau)]\, d\tau \\ &\quad + \sqrt{L} J'_\gamma(\omega L) \int_0^t w_n(L, \tau) \sin[\omega(t-\tau)]\, d\tau \\ &\quad - \frac{\sqrt{L}}{\omega} J_\gamma(\omega L) \int_0^t \left.\frac{\partial w_n(r, \tau)}{\partial r}\right|_{r=L} \sin[\omega(t-\tau)]\, d\tau. \end{aligned} \tag{12}$$

The last integral in (9) is the Hankel transform [19], it is inverse to itself:

$$\begin{aligned} \tilde{f}(\omega)\sqrt{\omega} &= \int_0^\infty f(r)\chi(r-L)\sqrt{r\omega} J_\gamma(\omega r)\, dr \\ &\leftrightarrow f(r)\chi(r-L) = \int_0^\infty \left(\tilde{f}(\omega)\sqrt{\omega}\right)\sqrt{r\omega} J_\gamma(\omega r)\, d\omega. \end{aligned}$$

Returning in (12) to the originals we obtain:

$$\begin{aligned} & w_n(r, t)\chi(r-L) \\ &= \int_0^\infty \tilde{w}_n(\omega, t)\omega\sqrt{r} J_\gamma(\omega r) d\omega = \int_0^t \left[\int_0^\infty J_\gamma(\omega r) J_\gamma(\omega L) \sin[\omega(t-\tau)]\, d\omega\right] \end{aligned}$$

$$\sqrt{r}\left[\frac{1}{2\sqrt{L}}w_n(L,\tau)-\sqrt{L}\,\frac{\partial w_n(r,\tau)}{\partial r}\bigg|_{r=L}\right]d\tau$$
$$+\int\limits_0^t\left[\int\limits_0^\infty \omega J_\gamma\left(\omega r\right)J_\gamma'\left(\omega L\right)\sin\left[\omega\left(t-\tau\right)\right]d\omega\right]\sqrt{rL}w_n\left(L,\tau\right)d\tau$$
$$=\int\limits_0^t I_1\left(r,L,t-\tau\right)\sqrt{r}\left[\frac{1}{2\sqrt{L}}w_n\left(L,\tau\right)-\sqrt{L}\,\frac{\partial w_n\left(r,\tau\right)}{\partial r}\bigg|_{r=L}\right]d\tau$$
$$+\int\limits_0^t I_2\left(r,L,t-\tau\right)\sqrt{rL}w_n\left(L,\tau\right)d\tau. \tag{13}$$

It is easy to verify (see [10, 17, 20]) that for $r>L>0$ we have

$$I_1(r,L,t-\tau)$$
$$=\begin{cases}0, & 0<t-\tau<r-L\\ \dfrac{1}{2\sqrt{rL}}P_{\gamma-1/2}\left(\dfrac{r^2+L^2-(t-\tau)^2}{2rL}\right), & r-L<t-\tau<r+L\\ -\dfrac{\cos\gamma\pi}{\pi\sqrt{rL}}Q_{\gamma-1/2}\left(-\dfrac{r^2+L^2-(t-\tau)^2}{2rL}\right), & t-\tau>r+L\end{cases}$$
$$=\frac{1}{2\sqrt{rL}}P_n(q)\chi\left[(t-\tau)-(r-L)\right]\chi\left[(r+L)-(t-\tau)\right],\quad t-\tau>0 \tag{14}$$

and

$$I_2(r,L,t-\tau)=\frac{\partial}{\partial L}I_1(r,L,t-\tau)$$
$$=\frac{1}{2\sqrt{rL}}\chi[(t-\tau)-(r-L)]\chi[(r+L)-(t-\tau)]\left[-\frac{P_n(q)}{2L}+\frac{P_n^1(q)}{\sqrt{1-q^2}}\left(\frac{1}{r}-\frac{q}{L}\right)\right]$$
$$+\frac{1}{2\sqrt{rL}}P_n\left(q\right)\left[\delta\left(t-\tau-r+L\right)+\delta\left(r+L-t+\tau\right)\right],\quad t-\tau>0 \tag{15}$$

Here, $q = [r^2+L^2-(t-\tau)^2](2rL)^{-1}$; $P_\gamma\left(x\right)$ and $Q_\gamma\left(x\right)$ are the Legendre functions of the first and second kind, respectively. We have also used the following properties of derivatives [18, 21]:

$$\frac{dP_n(q)}{dq}=\frac{1}{\sqrt{1-q^2}}P_n^1(q)\quad\text{and}\quad\frac{d\chi(x)}{dx}=\delta(x).$$

Thus, summarizing results (6), (7), and (13)–(15), we obtain

$$u_n(r,t) = \frac{L}{2r}\left\{\int\limits_{t-(r+L)}^{t-(r-L)}\left[\left(\frac{L-rq}{rL\sqrt{1-q^2}}P_n^1(q) - \frac{1}{L}P_n(q)\right)u_n(L,\tau)\right.\right.$$
$$\left.-P_n(q)\,\frac{\partial u_n(r,\tau)}{\partial r}\bigg|_{r=L}\right]d\tau + u_n\left(L, t-(r-L)\right) + (-1)^n$$
$$\left.u_n\left(L, t-(r+L)\right)\right\}, \quad r > L, \quad n = 1, 2, \ldots \tag{16}$$

and

$$U(r,\vartheta,t) =$$
$$\frac{L}{2r}U(L,\vartheta,t-(r-L)) + \frac{L}{2r}\sum_{n=1}^{\infty}\left\{\int\limits_{t-(r+L)}^{t-(r-L)}\left[\left(\frac{L-rq}{rL\sqrt{1-q^2}}P_n^1(q) - \frac{1}{L}P_n(q)\right)\right.\right.$$
$$\int\limits_0^{\pi} U(L,\vartheta_1,\tau)\mu_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1$$
$$\left.-P_n(q)\int\limits_0^{\pi}\frac{\partial U(r,\vartheta_1,\tau)}{\partial r}\bigg|_{r=L}\mu_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\right]d\tau$$
$$\left.+\int\limits_0^{\pi}(-1)^n U(L,\vartheta_1,t-(r+L))\mu_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\right\}\mu_n(\cos\vartheta),$$
$$0 \le \vartheta \le \pi\,, \quad r > L. \tag{17}$$

Formulas (16) and (17) represent the exact radiation condition for the pulsed waves generated by an axially-symmetrical unit and outgoing towards $r \to \infty$. Namely, formula (16) specifies behaviour of the space-time amplitudes of all partial components of these waves propagating in free space, whereas formula (17) describes these waves integrally. It is evident that formula (17) determines the transport operator $Z_{p\in\Gamma\to g\in\Omega_{ext}}(t)$ as well, which allows one to calculate the field at all points of Ω_{ext} including the points of the far-field zone from the values of the field $U(g,t)$ on the boundary Γ.

By passing to the limit $r \to L$ in (17), we obtain

$$U(L,\vartheta,t) = \sum_{n=1}^{\infty}\left\{\int\limits_{t-2L}^{t}\left[\left(\frac{t-\tau}{L\sqrt{4L^2-(t-\tau)^2}}P_n^1\left(1-\frac{(t-\tau)^2}{2L^2}\right)\right.\right.\right.$$

$$\left. -\frac{1}{L}P_n\left(1-\frac{(t-\tau)^2}{2L^2}\right)\right) \times \int\limits_0^{\pi} U\left(L,\vartheta_1,\tau\right)\mu_n\left(\cos\vartheta_1\right)\sin\vartheta_1 d\vartheta_1$$

$$\left. -P_n\left(1-\frac{(t-\tau)^2}{2L^2}\right)\int\limits_0^{\pi}\left.\frac{\partial U\left(r,\vartheta_1,\tau\right)}{\partial r}\right|_{r=L}\mu_n\left(\cos\vartheta_1\right)\sin\vartheta_1 d\vartheta_1\right] d\tau$$

$$\left. +(-1)^n\int\limits_0^{\pi} U(L,\vartheta_1,t-2L)\mu_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\right\}\mu_n(\cos\vartheta), \quad 0\leq\vartheta\leq\pi. \tag{18}$$

Formula (18) represents the exact absorbing condition on the artificial boundary Γ. This condition is spoken of as exact because any outgoing wave described by the initial problem (4) satisfies this condition. Every outgoing wave $U\left(g,\, t\right)$ passes through the boundary Γ without distortions, as if it is absorbed by the domain Ω_{ext} or its boundary Γ. That is why this condition is said to be absorbing.

4. EQUIVALENCE THEOREM

The boundary condition (18) allows us to replace the original open problem with the closed initial boundary value problem

$$\begin{cases} \left[-\varepsilon(g)\dfrac{\partial^2}{\partial t^2} - \sigma(g)\dfrac{\partial}{\partial t} + \dfrac{\partial^2}{\partial z^2} + \dfrac{\partial}{\partial \rho}\left(\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\rho\right)\right] U(g,t) \\ = F(g,t), \quad t>0, \quad g\in\Omega_{int} \\ U\left(g,t\right)\big|_{t=0} = \varphi(g), \quad \left.\dfrac{\partial}{\partial t}U(g,t)\right|_{t=0} = \psi(g), \quad g=\{\rho,z\}\in\overline{\Omega_{int}} \\ E_{tg}(p,t)\big|_{p=\{\rho,\phi,z\}\in\Sigma} = 0, \quad t\geq 0 \\ U(0,z,t)=0, \quad |z|<L, \quad t\geq 0 \\ M[U(g,t)]\big|_{g\in\Gamma} = 0, \quad t\geq 0. \end{cases} \tag{19}$$

(the operator M here is given by (18) with the subsidiary formula (17). Below we prove that this replacement is equivalent, in other words, that any solution to the problem (4) is at the same time the solution to the problem (17), (19), and vice versa.

Problem (4) is uniquely solvable in the space of generalized functions $\mathbf{W}_2^1(\Omega^T)$. Its unique solution $U(g,t)$ is at the same time a solution to problem (19) from the Sobolev space $\mathbf{W}_2^1(\Omega_{int}^T)$, $\Omega_{int}^T = \Omega_{int} \times (0,T)$. This direct inclusion is trivial. It is proved by the constructions from Section 2. The inverse inclusion will also be true if only the generalized solution $U(g,t)$ of problem (19) belonging to the space $\mathbf{W}_2^1(\Omega_{int}^T)$ is unique. Let us prove the uniqueness.

According to [16], the generalized solution of problem (19) is an element $U(g,t)$ of the space $\mathbf{W}_2^1(\Omega_{int}^T)$ being equal to $\varphi(g)$ at $t=0$ and satisfying the identity

$$\int\limits_{\Omega_{int}^T} \left[\varepsilon \frac{\partial U}{\partial t}\frac{\partial \gamma}{\partial t} - \left(\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\rho U\right)\frac{\partial(\rho\gamma)}{\partial \rho} - \frac{\partial U}{\partial z}\frac{\partial \gamma}{\partial z} - \sigma\frac{\partial U}{\partial t}\gamma\right] dgdt$$
$$+\int\limits_{\Phi^T} \left[\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho U\right)\gamma\cos(\vec{n},\vec{\rho}) + \frac{\partial U}{\partial z}\gamma\cos(\vec{n},\vec{z})\right] dsdt + \int\limits_{\Omega_{int}} \varepsilon\psi\gamma(g,0)dg$$
$$= \int\limits_{\Omega_{int}^T} F\gamma dgdt \tag{20}$$

for any function $\gamma(g,t)$ from $\mathbf{W}_2^1(\Omega_{int}^T)$ that is zero at $t=T$. Here Φ^T is a lateral surface of the cylinder Ω_{int}^T ($\Phi^T = \Phi \times (0,T)$; Φ is the boundary of the domain Ω_{int}); $\cos(\vec{n},\vec{\rho})$ and $\cos(\vec{n},\vec{z})$ are cosines of the angles between the outer normal $\vec{n}$ to the surface Φ^T and the axes $\vec{\rho}$ and $\vec{z}$, respectively. An element of the end surfaces of the cylinder is $dg = \rho d\rho dz$. Identity (20) is derived by multiplying the telegraph equation from (19) by $\gamma(g,t)$ and by integrating the result by parts in Ω_{int}^T [16, 22].

Assume that there exist two solutions of problem (19) belonging to the space $\mathbf{W}_2^1(\Omega_{int}^T)$: $U_1(g,t)$ and $U_2(g,t)$. The difference of these solutions $u(g,t) = U_1(g,t) - U_2(g,t)$ satisfies the identity

$$\int\limits_{\Omega_{int}^T} \left[\varepsilon \frac{\partial u}{\partial t}\frac{\partial \gamma}{\partial t} - \left(\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\rho u\right)\frac{\partial(\rho\gamma)}{\partial \rho} - \frac{\partial u}{\partial z}\frac{\partial \gamma}{\partial z} - \sigma\frac{\partial u}{\partial t}\gamma\right] dgdt$$
$$+\int\limits_{\Phi^T} \left[\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho u\right)\gamma\cos(\vec{n},\vec{\rho}) + \frac{\partial u}{\partial z}\gamma\cos(\vec{n},\vec{z})\right] dsdt = 0. \tag{21}$$

Let us introduce an arbitrary $\tau \in (0,T)$ and consider the following function

$$\gamma(g,t)=\begin{cases}\int\limits_t^\tau u(g,\zeta)d\zeta, & 0<t<\tau\\ 0, & \tau<t<T\end{cases}.$$

It can be verified easily that $\gamma\,(g,t)$ has in Ω_{int}^T the generalized derivatives

$$\frac{\partial\gamma(g,t)}{\partial t}=\begin{cases}-u(g,t), & 0<t<\tau\\ 0, & \tau<t<T\end{cases},\quad \frac{\partial\gamma(g,t)}{\partial\rho}=\begin{cases}\int\limits_t^\tau\frac{\partial u(g,\zeta)}{\partial\rho}d\zeta, & 0<t<\tau\\ 0, & \tau<t<T\end{cases},$$

$$\frac{\partial\gamma(g,t)}{\partial z}=\begin{cases}\int\limits_t^\tau\frac{\partial u(g,\zeta)}{\partial z}d\zeta, & 0<t<\tau\\ 0, & \tau<t<T\end{cases}.$$

At the same time we have $\gamma(g,t)|_{t=T}=0$. Substituting the function $\gamma\,(g,\,t)$ into identity (21), we obtain:

$$\int\limits_{\Omega_{int}^\tau}\left[\varepsilon\frac{\partial u}{\partial t}u+\frac{1}{\rho^2}\left(\frac{\partial}{\partial\rho}\rho u\right)\left(\int\limits_t^\tau\frac{\partial}{\partial\rho}\rho u(\zeta)d\zeta\right)+\frac{\partial u}{\partial z}\left(\int\limits_t^\tau\frac{\partial}{\partial z}u(\zeta)d\zeta\right)+\sigma\frac{\partial u}{\partial t}\gamma\right]dgdt$$
$$-\int\limits_{\Phi^\tau}\left[\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho u\right)\gamma\cos(\vec{n},\vec{\rho})+\frac{\partial u}{\partial z}\gamma\cos(\vec{n},\vec{z})\right]dsdt=0 \tag{22}$$

Since [22]

$$\int\limits_{\Omega_{int}^\tau}\left[k(g)f(g,t)\int\limits_t^\tau f(g,\zeta)d\zeta\right]dgdt=\frac{1}{2}\int\limits_{\Omega_{int}}k(g)\left(\int\limits_0^\tau f(g,t)dt\right)^2dg,$$

then

$$\int\limits_{\Omega_{int}^\tau}\left[\frac{1}{\rho^2}\left(\frac{\partial}{\partial\rho}\rho u\right)\left(\int\limits_t^\tau\frac{\partial}{\partial\rho}\rho u(\zeta)d\zeta\right)\right]dgdt=\frac{1}{2}\int\limits_{\Omega_{int}}\frac{1}{\rho^2}\left(\int\limits_0^\tau\frac{\partial}{\partial\rho}\rho udt\right)^2dg\geq 0 \tag{23}$$

and

$$\int\limits_{\Omega_{int}^\tau}\left[\frac{\partial u}{\partial z}\left(\int\limits_t^\tau\frac{\partial}{\partial z}u\,(\zeta)\,d\zeta\right)\right]dgdt=\frac{1}{2}\int\limits_{\Omega_{int}}\left(\int\limits_0^\tau\frac{\partial}{\partial z}udt\right)^2dg\geq 0. \tag{24}$$

By performing partial integration and taking into consideration that $\gamma(g,t)|_{t=\tau}=0$ and $u(g,t)|_{t=0}=0$, we also obtain

$$\int\limits_{\Omega_{int}^{\tau}} \left[\varepsilon \frac{\partial u}{\partial t} u\right] dgdt = \frac{1}{2} \int\limits_{\Omega_{int}} \varepsilon \left[u\left(g,\tau\right)\right]^2 dg \geq 0 \tag{25}$$

and

$$\int\limits_{\Omega_{int}^{\tau}} \left[\sigma \frac{\partial u}{\partial t} \gamma\right] dgdt = -\int\limits_{\Omega_{int}^{\tau}} \left[\sigma u \frac{\partial \gamma}{\partial t}\right] dgdt = \int\limits_{\Omega_{int}^{\tau}} \sigma u^2 dgdt \geq 0. \tag{26}$$

Thus all the volume integrals entering identity (22) are nonnegative. Show that the integral

$$I_3\left(\tau\right) = -\int\limits_{\Phi^{\tau}} \left[\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho u\right)\gamma \cos\left(\vec{n},\vec{\rho}\right) + \frac{\partial u}{\partial z}\gamma \cos\left(\vec{n},\vec{z}\right)\right] dsdt \tag{27}$$

is nonnegative as well. To this end, let us estimate the integral $I_3\left(\tau\right)$ for the case of TE_{0n}-waves, when [9]

$$\begin{aligned} u(g,t) &= E_{\phi}, \quad E_{\rho} = E_z = H_{\phi} \equiv 0 \\ \text{and} \quad \frac{\partial H_{\rho}}{\partial t} &= \eta_0^{-1} \frac{\partial u}{\partial z}, \quad \frac{\partial H_z}{\partial t} = -\eta_0^{-1} \frac{1}{\rho}\frac{\partial (\rho u)}{\partial \rho} \end{aligned} \tag{28}$$

(the case of TM_{0n}-waves can be considered similarly).

$$\begin{aligned} I_3\left(\tau\right) &= \eta_0 \int\limits_{\Phi^{\tau}} \left[\frac{\partial H_z}{\partial t}\gamma \cos\left(\vec{n},\vec{\rho}\right) - \frac{\partial H_{\rho}}{\partial t}\gamma \cos\left(\vec{n},\vec{z}\right)\right] dsdt \\ &= -\eta_0 \int\limits_{\Phi^{\tau}} \left[H_z \frac{\partial \gamma}{\partial t} \cos\left(\vec{n},\vec{\rho}\right) - H_{\rho} \frac{\partial \gamma}{\partial t} \cos\left(\vec{n},\vec{z}\right)\right] dsdt \\ &= \eta_0 \int\limits_{\Phi^{\tau}} \left[H_z u \cos\left(\vec{n},\vec{\rho}\right) - H_{\rho} u \cos\left(\vec{n},\vec{z}\right)\right] dsdt \\ &= \eta_0 \int\limits_{\Phi^{\tau}} \left[H_z E_{\phi} \cos\left(\vec{n},\vec{\rho}\right) - H_{\rho} E_{\phi} \cos\left(\vec{n},\vec{z}\right)\right] dsdt \\ &= \eta_0 \int\limits_{\Gamma\times(0,\tau)} \left[H_z E_{\phi} \cos\left(\vec{n},\vec{\rho}\right) - H_{\rho} E_{\phi} \cos\left(\vec{n},\vec{z}\right)\right] dsdt \\ &= \eta_0 \int\limits_{\Gamma\times(0,\tau)} \left(\left[\vec{E}\times\vec{H}\right]\cdot\vec{n}\right) dsdt = \eta_0 I_4\left(\tau\right) \geq 0. \end{aligned} \tag{29}$$

The last step in the chain of transformations (29) requires explanation. The integral $I_4(\tau)$, accurate within a fixed factor, coincides with the electromagnetic field energy radiated from the region $\Omega_{int} \times [0 \leq \varphi \leq 2\pi]$ during the time $0 < t < \tau$ [23]. According to the condition $M[u(g,t)]|_{g\in\Gamma} = 0$, the functions $\vec{E} = \{E_\rho, E_\phi, E_z\}$ and $\vec{H} = \{H_\rho, H_\phi, H_z\}$ correspond to the electromagnetic waves outgoing from the domain Ω_{int}; and the energy of the outgoing waves cannot be negative.

Then, from (22)–(26), (29) we have

$$\int\limits_{\Omega_{int}} \varepsilon \left[u\left(g,\tau\right)\right]^2 dg = \int\limits_{\Omega_{int}^{\tau}} \sigma u^2 dg dt = 0,$$

or, in view of arbitrariness of τ,

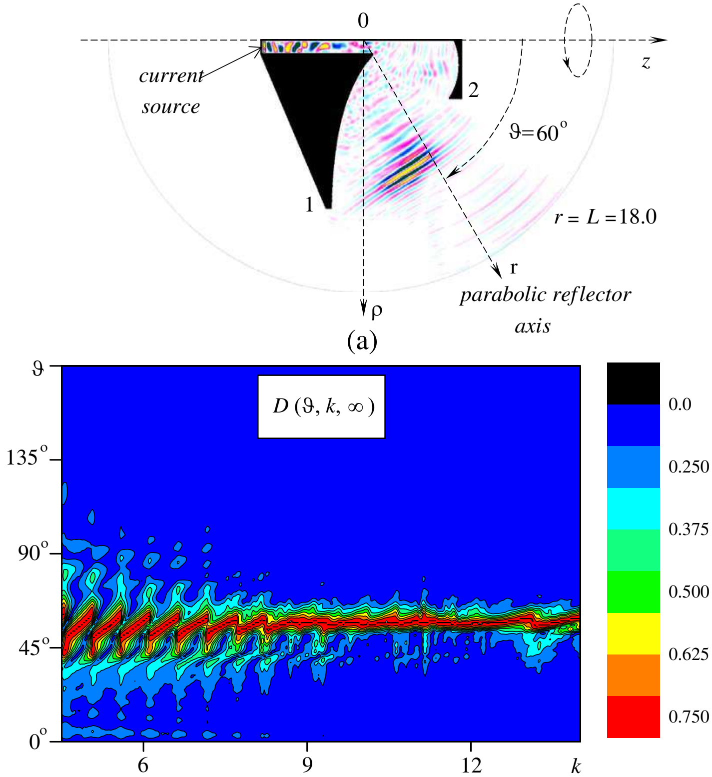
$$u\left(g,t\right) \equiv 0, \quad g \in \Omega_{int}, \quad 0 < t < T.$$

Statement 2. *Let problem (4) has a unique solution from* $\mathbf{W}_2^1\left(\Omega^T\right)$. *Then, problem (19) is uniquely solvable in the space* $\mathbf{W}_2^1\left(\Omega_{int}^T\right)$, *and closed problem (17), (19) is equivalent to open problem (4).*

5. NUMERICAL EXAMPLE

Examples of numerical implementation of the approach based on the application of the exact absorbing conditions in standard finite-difference algorithms can be found in our previous papers [11, 12, 14, 24, 25] on physical analysis of various open electrodynamic structures. The analytical results presented above as well as in our paper [10] have been realized in software intended for calculation of temporal and frequency electrodynamic characteristics of open resonators, microwave energy compressors, and omnidirectional radiators of pulsed and monochromatic signals. By way of illustration, we present below some numerical results obtained for omnidirectional reflector antennas that operate in the frequency range, where methods of physical and geometrical optics are inapplicable. Geometry of these antennas (as a first approximation) can be constructed in the same way as in quasi-optical case, by using the well-known ellipse, hyperbola, and parabola focal properties.

In [26], a scheme for constructing omnidirectional reflector antennas efficiently radiating TM_{0n}- and TE_{0n}-waves in the given direction $\vartheta \in (30°, 90°)$ is presented. That technique is based on the following facts (see Fig. 2(a)). An incident wave (the wave incoming from a circular or coaxial waveguide) passes through the focus of the upper elliptic mirror 2 and is directed through the second focus onto



(b)

Figure 2. Excitation of the axially-symmetric perfectly conducting antenna by a pulsed TE_{0n}-wave, (a) geometry of the antenna (the drawing is in proportion) and the field pattern for $E_\phi(g, t)$ at the instant the principal part of the pulse propagates within the computational domain Ω_{int} ($t = 55$) with the artificial boundary $\Gamma = \{g = \{r, \vartheta\} \in \Omega\colon\ r = L = 18.0\}$, (b) the directional pattern $D(\vartheta, k, \infty)$ in the frequency band $4.5 \leq k \leq 14$.

the principal, parabolic mirror 1. If the second focus of the auxiliary reflector 2 is coincident with the focus of the principal mirror 1, then the propagation direction of the radiated wave must coincide with the axis of the parabolic reflector. Our approach allows the geometry of the antennas of this kind to be optimized with regard to the requirements on their energy (radiation efficiency) and spatial (radiation directivity) characteristics.

To illustrate, in Fig. 2 the numerical results are presented for the case, where the pulsed TE_{0n}-waves generated by a current source

$F(g, t)$ located near a back wall of a short-circuited coaxial waveguide excite the reflector antenna. The radius of antenna's elliptic reflector 2 is 4.2, while the eccentricity equals 0.37. The radius of parabolic reflector 1 is equal to 12.0, its focal length is 2.88, and the angle between the parabola axis and z-axis is 60°.

The time-dependence of the current source at all its points g is given by the function $F(g,t) = 4\sin\,[\Delta k(t-\tilde{T})]\cos[\tilde{k}(t-\tilde{T})](t-\tilde{T})^{-1}$ $\chi(\bar{T}-t)$, where the delay time $\tilde{T} = 25$, the central frequency $\tilde{k} = 9.25$, the efficient duration $\bar{T} = 50$, and the parameter $\Delta k = 4.75$ are such that the generated pulsed electromagnetic wave occupies the frequency band $4.5 \leq k \leq 14.0$. Within this frequency range, the feeding coaxial waveguide with the outer conductor of radius $a = 1.0$ and the inner conductor of radius $b = 0.1$ sustains propagation of one ($k < k_2 \approx 7.33$), two ($k_2 < k < k_3 \approx 10.75$), or three ($k_3 < k$) TE_{0n}-waves. Here, $k = 2\pi/\lambda$ is the wave number (frequency parameter or simply frequency), λ is the wavelength, k_n is the cutoff point for the TE_{0n}-wave in the waveguide. The space-frequency characteristics $\tilde{f}(g, k)$ are obtainable from the time-frequency characteristics $f(g, t)$ by applying the Fourier transform

$$f(g,t) \quad \leftrightarrow \quad \tilde{f}(g,k) = \frac{1}{2\pi}\int\limits_0^T f(g,t)\,\mathrm{e}^{ikt}dt,$$

where $t \in [0, T]$ is the observation time interval; the functions $f(g, t)$ are assumed equal to zero for $t > T$.

In the directional pattern $D(\vartheta, k, \infty)$, a single main lobe directed at the angle $\bar{\vartheta}(k) \approx 57.5°$ dominates in the range $k > 8$ ($\lambda < 0.78$) (Fig. 2(b)). This angle differs slightly from the axis of the parabolic reflector 1. This distinction is caused basically by the fact that the antinodes of the wave beam travelling from the input waveguide onto elliptic mirror 2 do not coincide exactly with its first focus. Here, $D(\vartheta, k, M) = |\tilde{E}_\varphi(r = M, \vartheta, k)|^2 / \max\limits_{0<\vartheta<\pi} |\tilde{E}_\varphi(r = M, \vartheta, k)|^2$ is the normalized power pattern on the arc $r = M \geq L$ ($0 \leq \vartheta \leq 180°$). The main lobe is directed at the angle $\bar{\vartheta}(k)$: $D(\bar{\vartheta}(k), k, M) = 1$.

Note also that the possibility to compute directional patterns within sufficiently wide frequency bands (see, for example, Fig. 2(b)) as well as to study time transformations of the field at $\vartheta \approx \bar{\vartheta}$ are of considerable value in designing ultra-wide band omnidirectional communication antennas that retain their directional properties within the range and provide a faithful reproduction of the excitation.

6. CONCLUSION

In the paper, novel results associated with the construction of rigorous models of nonsine electrodynamics have been obtained by developing the technique previously used in classical works for studying initial boundary value problems with Dirichlet and Neumann boundary conditions (see, for example [16, 22]). For an open initial boundary value problem describing transient states of the field in axially-symmetrical compact electrodynamic structures, the exact absorbing boundary conditions have been derived, which allows one to truncate the domain of analysis down to a bounded region. The equivalency of the modified closed problem and the original open initial boundary value problem has been proved. A numerical example is presented to exemplify implementation of the above approach.

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