The Merits of the Integral Equation Methods in Computational Optoelectronicsics

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2. Fundamentals of Analytical Regularization

Reduction of EM BVP to a Fredholm's operator equation

Boundary-value problem

- **Maxwell equations**
- **Boundary conditions**
- **Edge condition**
- **Radiation condition**

⇒

- **Integral equations**
- **Series equations**

$$GX = Y$$

⇓

$$G = G_1 + G_2$$

Suppose that:

G_1 **is more singular than** G_2

and G_1^{-1} **is known**

⇐

$$X + AX = B$$

$$A = G_1^{-1} G_2, \quad B = G_1^{-1} Y$$

$$\| A \|_{L_2} < \infty, \quad B \in L_2$$

Fredholm 2-nd kind IE or infinite-matrix equation

3. Consequences of Regularization

Fredholm's theorems

$$X + A(k)X = B$$

Existence of exact solution:

If operator equation is equivalent to BVP, then its solution is unique for all real wavenumbers *k* ⟹ $X = (I + A(k))^{-1} B$

Point-wise convergence of discrete solutions:

$$e(N) = \| X - X^N \| (\| X \|)^{-1}$$
$$\leq \| (I + A)^{-1} \| \cdot \| A - A^N \| \xrightarrow[N \to \infty]{} 0$$

Condition number is stable:

$$cond(I + A) = \| I + A \| \cdot \| (I + A)^{-1} \| < \infty$$

4. Efficient Regularization Schemes

What is invertible?

$$G \Longrightarrow G = G_1 + G_2 \quad ?G_1$$

- **Canonical-shape part (circular-cylinder & sphere)**

  well developed – trigonometric basis; used for multiple canonical scatterers & in a layered host medium

Small-contrast part

  well developed – Muller equations, (loaded) volume IE

- **Static part: PEC and imperfect zero-thickness screens**

  well developed – EFIE + Chebyshev basis in 2-D;

  variants – in FT domain, RHP (in periodic case)

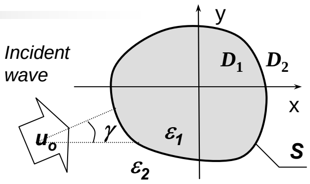
- **HF (halfplane) part: PEC and imperfect screens**

  scarcely developed – most promising for solving big problems of quasioptics with economic algorithms

5. Dielectric Cylinder Scattering - Formulation

Boundary-value problem

Scattering by an arbitrary smooth dielectric cylinder. Incident field is a plane wave in the reception mode and a directive localized source field in the transmission mode



wavenumber $k_j = k\sqrt{\varepsilon_j \mu_j}$ $\quad \alpha_j = 1/\mu_j$ or $1/\varepsilon_j$

2D BVP: **Find** $u_j(\vec{r}),\quad r \in D_j$ j=1,2, **such that**

1. Helmholtz equation off S: $(\Delta + k_j^2) u_j^{sc}(\vec{r}) = 0$

2. Boundary conditions at S:

$$u_1(\vec{r})\big|_S = u_2(\vec{r})\big|_S \quad \textbf{and} \quad \alpha_1 \frac{\partial u_1(\vec{r})}{\partial n}\bigg|_S = \alpha_2 \frac{\partial u_2(\vec{r})}{\partial n}\bigg|_S$$

3. Sommerfeld radiation condition

6. Muller's Boundary Integral Equations

Small contrast inversion

Fields representation = combination of the single and double layer potentials

$$u_1(\vec{r}) = \int_S \left[p_1(\vec{r}_s) \frac{\partial G_1(\vec{r},\vec{r}_s)}{\partial n_s} - q_1(\vec{r}_s) G_1(\vec{r},\vec{r}_s) \right] dl_s \qquad \vec{r} \in D_1$$

$$j = 1,2$$

$$u_2(\vec{r}) = \int_S \left[q_2(\vec{r}_s) G_2(\vec{r},\vec{r}_s) - u_2(\vec{r}_s) \frac{\partial G_2(\vec{r},\vec{r}_s)}{\partial n_s} \right] dl_s + u_0(\vec{r}) \qquad \vec{r} \in D_2$$

Parameterization + Boundary conditions =>

Uniquely solvable set of BIEs of the Fredholm 2nd kind :

$$\begin{cases} p_1(t) - \int_0^{2\pi} p_1(t_s) A(t,t_s)\, dt_s + \int_0^{2\pi} q_1(t_s) B(t,t_s) dt_s = L(t)\, u_0(t) \\ \left(1 + \dfrac{\alpha_1}{\alpha_2}\right) \dfrac{q_1(t)}{2} - \int_0^{2\pi} p_1(t_s) C(t,t_s)\, dt_s + \int_0^{2\pi} q_1(t_s) D(t,t_s) dt_s = L(t) \dfrac{\partial u_0(t)}{\partial n} \end{cases}$$

$\alpha_j = \mu_j$ or ε_j for E- or H-polarization

7. Muller's Boundary Integral Equations

Kernel properties & discretization

In one of the kernels, a *log*-type singularity is kept; others are regular

$$A(t,t_s)=L(t)\left(\frac{\partial G_1}{\partial n_s}-\frac{\partial G_2}{\partial n_s}\right)\quad B(t,t_s)=L(t)\left(G_1-\frac{\alpha_1}{\alpha_2}G_2\right)\quad C(t,t_s)=L(t)\left(\frac{\partial^2 G_1}{\partial n_s\partial n}-\frac{\partial^2 G_2}{\partial n_s\partial n}\right)$$

$\|A\|, \ldots, \|D\| < (\varepsilon-1)$*Const*

$$D(t,t_s)=L(t)\left(\frac{\partial G_1}{\partial n}-\frac{\alpha_1}{\alpha_2}\frac{\partial G_2}{\partial n}\right)$$

Computing the singular integrals is improved by adding and subtracting the canonical-circle operators, e.g. G^0

$$\overset{o}{G}=\frac{i}{4}H_0\left(2kaSin\left|(t-t_s)/2\right|\right)$$

MBIEs + trigonometric-Galerkin discretization
=>Fred.-2 Matrix Equation filled in with DFFT

$$p(t)L(t)=\frac{2}{i\pi}\sum_{m=-\infty}^{\infty}p_n e^{imt}$$

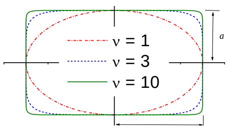
$$\begin{cases}\sum_{m=-\infty}^{\infty}p_m(\delta_{km}+A_{km})+\sum_{m=-\infty}^{\infty}q_m B_{km}=u_k\\ \sum_{m=-\infty}^{\infty}p_m C_{km}+\sum_{m=-\infty}^{\infty}q_m(\delta_{km}+D_{km})=\bar{u}_k\end{cases}$$

If the natural parameterization is used: $L(t)=1$

8. MBIE Algorithm Properties

Test example: super-ellipse

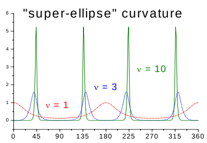
"super-ellipse" cross-section



Homogeneous dielectric cylinder:

«super-ellipse» = rectangle with smoothed edges:

$$(x/la)^{2\nu}+(y/a)^{2\nu}=1\qquad 0<\nu<\infty$$



Relative computational error, determined by norm in l_2

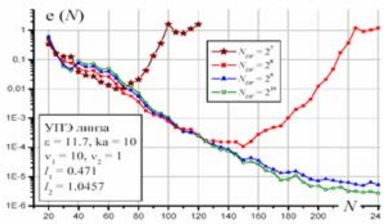
$$e(N)=\frac{\left\|Z^N-Z^{N+1}\right\|}{\left\|Z^N\right\|}$$

where

$$Z^N=\left\{z_n^{1N},z_n^{2N}\right\}$$

3 digit accuracy is achieved if

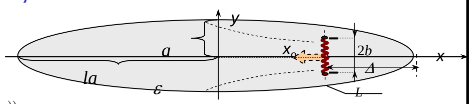
$$N\approx kal\sqrt{\varepsilon}+\nu+10$$



Computational error versus matrix block size N

9. Radiation of a Dielectric Rod Antenna

Far-field characteristics: pattern, radiated power & directivity

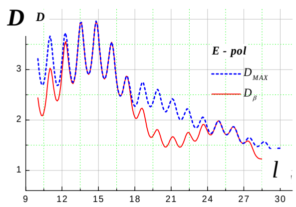


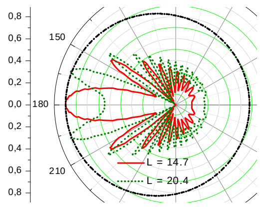
$$\Phi(\varphi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}p_n\int_0^{2\pi}e^{-ik(\cos(\varphi)x(t_s)+\sin(\varphi)y(t_s))}\cdot e^{int_s}dt_s$$

$$P^{rad}=\alpha_0\frac{2}{\pi}\int_0^{2\pi}|\Phi(t)|^2dt$$

$$D(\varphi)=2\alpha_0|\Phi(\varphi)|^2\cdot\left(kP^{rad}\right)^{-1}$$

Elliptic rod geometry and notations:
curvy line is the CSP feed aperture,
"l" is the elongation (axes ratio)





***Directivity versus the rod elongation;* dashed = in the main lobe, solid = along the axis ($\varphi=\beta$). ka =1, ε =2.5, ν =1, kb =0.1, $\beta=\pi$, Δ/a=0.7; y_0/a =0.**

Normalized Radiation Pattern

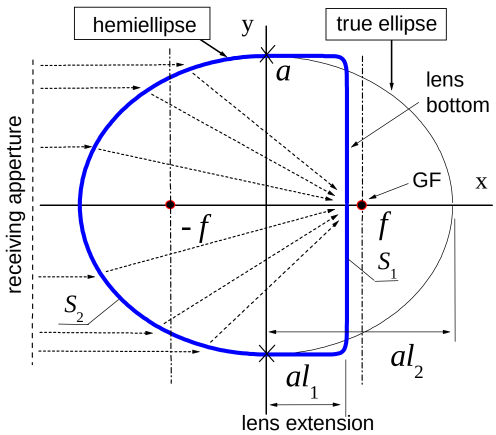
10. Wave Focusing by Elliptic-Front Lenses

Geometry & Principle

Cross-sectional contour: $S=S_1\cup S_2$

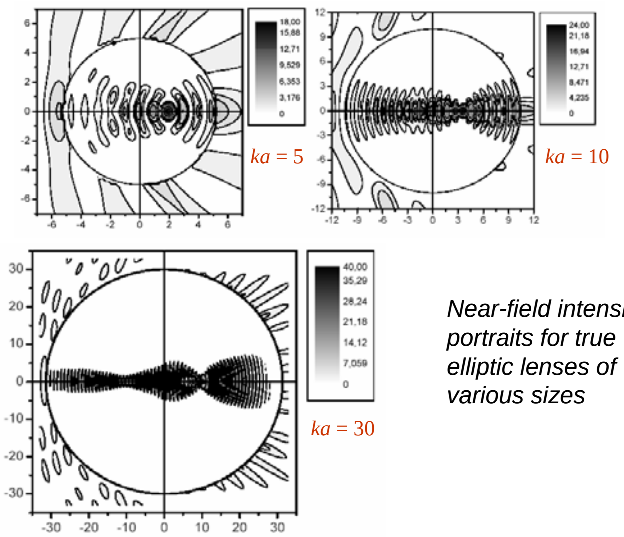
twice-continuous curve combined from **smoothly joined halves** **of ellipse and super-ellipse**

GO: Parallel rays come to the rear focus of the ellipse if the eccentricity is $e=1/\sqrt{\varepsilon}$



11. Wave Focusing by Elliptic Lenses

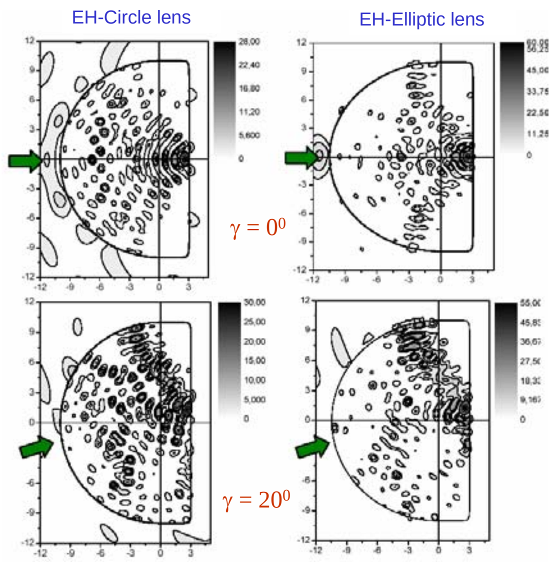
Near field characteristics



Near-field intensity portraits for true elliptic lenses of various sizes

12. Wave Focusing by Elliptic-Front Lenses

Near field characteristics



Near-field intensity portraits for

Extended

Hemi-Elliptic (EHE)

and

Hemi-Circular (EHC)

silicon lenses

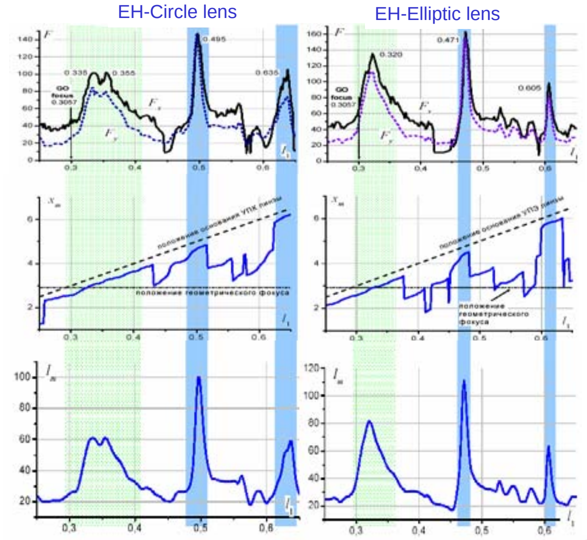
ka = 10

$\varepsilon = 11.7$

13. Wave Focusing by Elliptic-Front Lenses

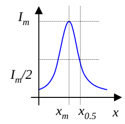
Near field characteristics

EH-Circle lens | EH-Elliptic lens

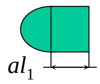


Focusability vs l_1

$$F_{x,y} = \frac{aI(x_m, y_m)}{|x, y_m - x, y_{0.5}|}$$



Main focus x-coordinate vs l_1



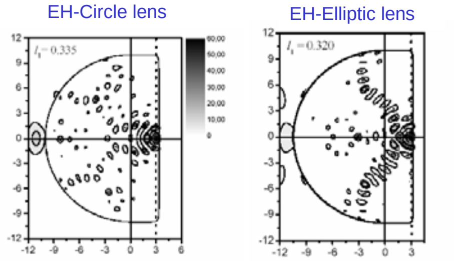
max intensity vs l_1

$\varepsilon = 11.7,\ ka = 10,\ l_2 = 1.0457,\ \nu_1 = 10,\ \nu_2 = 1,\ \gamma = 0^0,\ N_{DF} = 2^9$

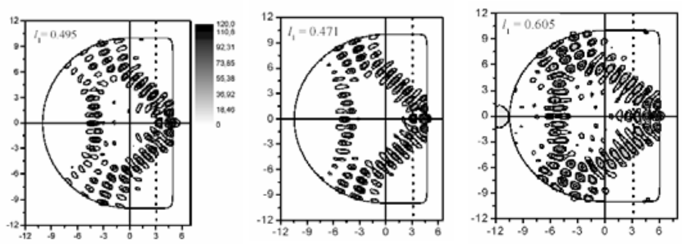
14. Wave Focusing by Elliptic-Front Lenses

Near field characteristics

EH-Circle lens | EH-Elliptic lens



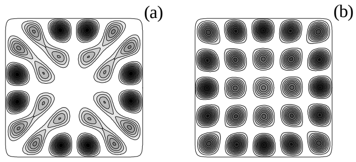
Near-field intensity portraits for EHE and EHC silicon lenses for various values of extension parameter (l_1)



$\varepsilon = 11.7,\ ka = 10,\ l_2 = 1.0457,\ \nu_1 = 10,\ \nu_2 = 1,\ \gamma = 0^0,\ N_{DF} = 2^{10}$.

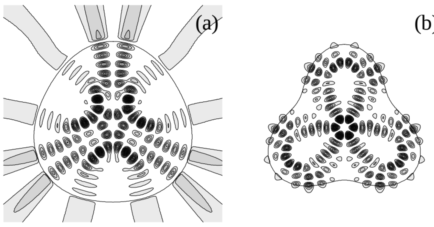
15. Natural Modes of Microcavities

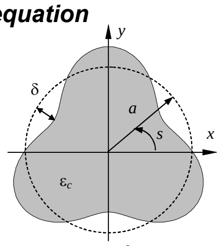
Near fields of modes in open 2-D dielectric resonators



Eigenfrequency boundary-value problem => Muller's boundary integral equations => analytical extraction of circular-contour part => determinant equation

***Det* (I+A(k))=0**





Modes in a square cavity with rounded edges (superellipse with ν**=10*)***

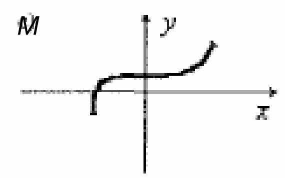
and in a curved triangular cavity

***Refractive index is* 2.63**

16. PEC Strip Scattering - Formulation

Boundary-value problem

Scattering by an arbitrary smooth open cylindrical PEC strip. Incident field is a plane wave in the RCS analysis and a directive localized feed field in the reflector antenna analysis



2-D BVP: find such $u^{sc}(\vec{r}), \quad u = u^{in} + u^{sc}$ **that**

1. Helmholtz equation off *M*: $(\Delta + k^2) u^{sc}(\vec{r}) = 0$
2. Boundary conditions at *M*:

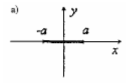
$$u(\vec{r})\big|_M = 0, \quad E - pol. \qquad \left.\frac{\partial u(\vec{r})}{\partial n}\right|_M = 0, \quad H - pol.$$

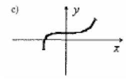
3. Sommerfeld radiation condition

17. Electric-Field Integral Equations

Singular Integral Equations

Fields representation – **single/double layer potential**:





E-pol. *H-pol.*

$$u(\vec r)=\int_M p(\vec r_s)G_0(\vec r,\vec r_s)dl_s,\quad u(\vec r)=\int_M q(\vec r_s)\frac{\partial}{\partial n_s}G_0(\vec r,\vec r_s)dl_s$$

$$G_0(\vec r,\vec r_s)=\frac{i}{4}H_0^{(1)}(k|\vec r-\vec r_s|)$$ *Free space Green's function*

Parameterization of contour *M* => Boundary conditions => **Singular IEs of the 1st kind**

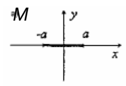
E-polarization *H-polarization*

$$\int_{-1}^{1}p(t_s)G_0(t,t_s)L(t_s)dt_s=-u^{in}(t),\quad \frac{\partial}{\partial n}\int_{-1}^{1}q(t_s)\frac{\partial}{\partial n_s}G_0(t,t_s)L(t_s)dt_s=-\frac{\partial u^{in}(t)}{\partial n}$$

Direct discretization of SIE does not guarantee convergence, is inefficient and inaccurate

18. EFIE & Method of Analytical Preconditioning

Static-part inversion: diagonalization with eigenfunctions



*Green's function decomposition =**static singular** part + regular part*

$$G_0(\vec r,\vec r_s)=\frac{i}{4}\ln|\vec r-\vec r_s|+R(k|\vec r-\vec r_s|)$$

E-pol.

$$\int_{-1}^{1}\frac{T_n(t_s)}{(1-t_s)^{1/2}}\ln(t-t_s)dt_s=\sigma_n T_n(t),\quad T_n(t)=$$ Chebyshev polynomials of the 1-st kind

H-pol.

$$\int_{-1}^{1}(1-t_s)^{1/2}U_n(t_s)\left.\frac{\partial^2\ln(\vec r-\vec r_s)}{\partial n^2}\right|_M dt_s=\tau_n U_n(t),\quad U_n(t)=$$ Chebyshev polynomials of the 2-nd kind

To transform SIE to the Fredholm 2nd kind matrix equation, take full set of the corresponding polynomials as a basis (I.e., make analytical preconditioning)

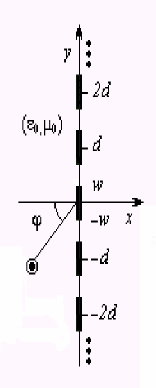
A X = B

⇓

X + CX = D

19. Scattering by a Dielectric Strip Grating

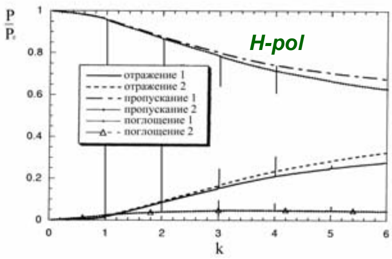
Static-part inversion: diagonalization with eigenfunctions



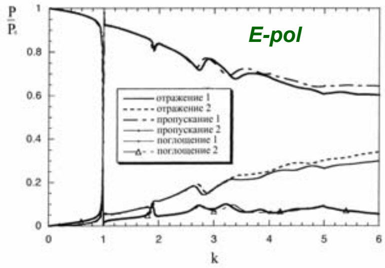
*Generalized boundary conditions on **thin dielectric strips**:*

$$(\vec{E}_T^+ + \vec{E}_T^-) = 2R(\vec{H}_T^+ - \vec{H}_T^-), \quad (\vec{H}_T^+ + \vec{H}_T^-) = 2Q(\vec{E}_T^+ - \vec{E}_T^-)$$

$$\text{where} \quad R = Q\zeta_r^2 = (i/2)\zeta_0\zeta_r \,\mathrm{ctg}(\sqrt{\varepsilon_r\mu_r}\,k_0\tau/2)$$



Fractions of transmitted, reflected and absorbed power versus normalized frequency, ε_r=10+i, φ=0^0, 2w/d=0.5, τ/d=0.01.

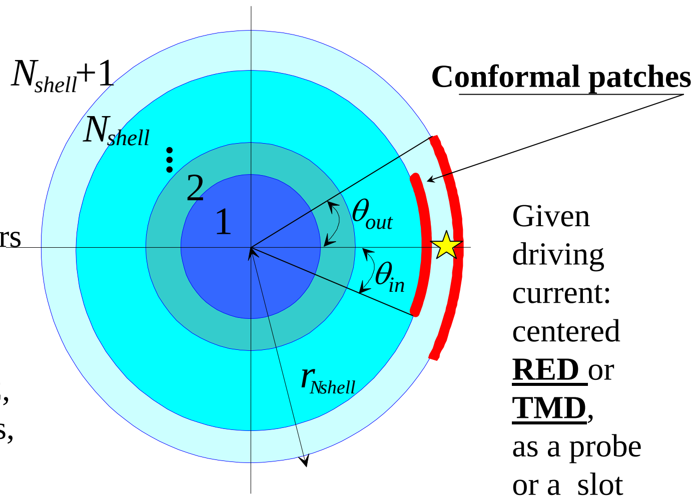


Fractions of transmitted, reflected and absorbed power versus normalized frequency, ε_r=10+i, φ=0^0, 2w/d=0.5, τ/d=0.01.

20. Radiation of Discrete Luneburg Lens Fed by Conformal Feed

Lens: concentric spherical layers of uniform dielectrics

Patches: PEC, zero-thickness, co-axial spherical disks, $0 \le \theta \le \pi$



$$\varepsilon(r) = 2 - \left(\frac{r}{R}\right)^2$$

Given driving current: centered **RED** or **TMD**, as a probe or a slot model

21. Radiation of Discrete Luneburg Lens Fed by Conformal Feed

SCMA structure :

$\theta_{in} = 0.02°$,
$\theta_{out} = 0.04°$

$r_{in}/r_{out} = 0.999$

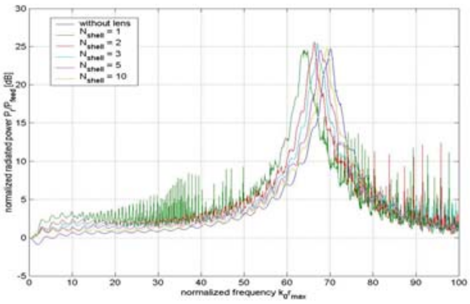
"large lens"

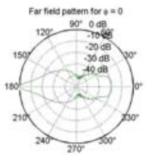
Lens structure :

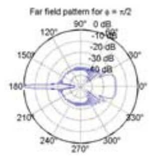
N_{shell} **: number of shells**

$\theta_i = 0$, $r_i = i/ N_{shell}$

$\varepsilon_{ri} = 2 - [(2i-1)/ 2N_{shell}]^2$
$\forall i,\ 1 \le i \le N_{shell}$







Lens => small shift of SCMA resonance + whispering-gallery modes

22. Conclusions: Merits of Analytical Regularization of Integral Equations

1. **Generates convergent and economic scattering algorithms with easily controlled accuracy**
2. **Leads to reliable simulations that predict even finest field features**
3. **Easily accesses quasioptical range**
4. **Is promising for CG iterative solvers**
5. **Can serve as a fast core for optimization**
6. **Enables explicit asymptotic solutions**
7. **Reduces eigenvalue problems to favorable determinant equations**