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Rigorous Formulation of the Lasing Eigenvalue Problem as a Spectral Problem for a Fredholm Operator Function

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Abstract—We propose a new convenient for mathematical investigation formulation of the lasing eigenvalue problem as a spectral problem for an operator-valued function, which involves boundary integral operators. We prove that these integral operators are weakly singular and the operator of the problem is Fredholm with index zero.

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1. INTRODUCTION

Various two-dimensional (2D) microcavity lasers have been investigated numerically with the aid of a modified electromagnetic eigenvalue problem, specifically tailored to extract the threshold values of gain in addition to the emission frequencies (see, for example, [15, 18, 19], and references therein). Such a modified formulation called the Lasing Eigenvalue Problem (LEP) was first introduced in 2004 in [11] and since then has gained credit in the photonics community. The greatest progress may have been achieved for two-dimensional microcavities with uniform gain in [13], where the original problem was reduced equivalently to a nonlinear spectral problem for the system of Muller boundary integral equations (BIEs), which was solved accurately by the Nystrom method. Derived first by Muller [10] this system has become a reliable and efficient tool for analysis of the electromagnetic field in the presence of a 2D homogeneous dielectric object with an arbitrary smooth boundary. Particularly, Muller BIEs were used for computations of eigenmodes of fully active [13] and passive microcavities [2, 3]. The original problem for microcavities with active regions have been also reduced recently to the system of Muller BIEs [14]. Numerical and theoretical investigations of microcavities with active regions are very important [12], but such studies have not been carried out in sufficient detail by rigorous mathematical methods.

In this paper we propose a new formulation of LEP for microcavities with active regions as a nonlinear spectral problem for a fredholm operator-valued function, which involves boundary integral operators. In Section 2 we describe the nonlinear spectral problem for the system of Muller BIEs constructed in [14]. In Section 3 we prove that all the boundary integral operators are weakly singular or have smooth kernels (Lemmas 1–4). It follows from these lemmas that the operator of the problem has the form $I - B$, where the operator B is compact (Theorem 1, Section 4) and I is the identical operator in the space of continuous functions. Obtained formulation is convenient for future study of the problem on the base of fundamental results of the theory of operator-valued functions in a pair of Banach spaces (see, for

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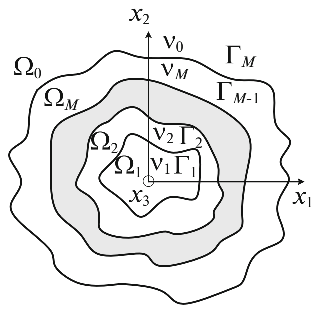

Fig. 1. Geometry of a 2D dielectric resonator with active zones.

example, [7, 8]) and the theory of continuous dependence of eigenvalues of operator-valued functions on real parameters developed in [16]. It also enables to apply the general results of the theory of approximation in nonlinear eigenvalue problems [5, 6] to a numerical analysis of the proposed in [13] and generalized in [14] computational algorithm. A similar approach was applied to spectral problems of the theory of dielectric waveguides [4].

2. MULLER BOUNDARY INTEGRAL EQUATIONS

The original problem was reduced in [14] to the following nonlinear with respect to the parameters k and γ eigenvalue problem for the system of Muller boundary integral equations:

$$u_1(x) - \int\limits_{\Gamma_1} K_1^{(1,3)}(k,\gamma;x,y)\,u_1(y)dl(y) - \int\limits_{\Gamma_1} K_1^{(1,4)}(k,\gamma;x,y)\,v_1(y)dl(y)$$

$$- \int\limits_{\Gamma_2} K_1^{(1,5)}(k,\gamma;x,y)\,u_2(y)dl(y) - \int\limits_{\Gamma_2} K_1^{(1,6)}(k,\gamma;x,y)\,v_2(y)dl(y) = 0, \quad x \in \Gamma_1, \tag{1}$$

$$u_m(x) - \int\limits_{\Gamma_{m-1}} K_m^{(1,1)}(k,\gamma;x,y)\,u_{m-1}(y)dl(y) - \int\limits_{\Gamma_{m-1}} K_m^{(1,2)}(k,\gamma;x,y)\,v_{m-1}(y)dl(y)$$

$$- \int\limits_{\Gamma_m} K_m^{(1,3)}(k,\gamma;x,y)\,u_m(y)dl(y) - \int\limits_{\Gamma_m} K_m^{(1,4)}(k,\gamma;x,y)\,v_m(y)dl(y)$$

$$- \int\limits_{\Gamma_{m+1}} K_m^{(1,5)}(k,\gamma;x,y)\,u_{m+1}(y)dl(y) - \int\limits_{\Gamma_{m+1}} K_m^{(1,6)}(k,\gamma;x,y)\,v_{m+1}(y)dl(y) = 0, \tag{2}$$

where $x \in \Gamma_m$, $m = 2, 3, \ldots, M-1$,

$$u_M(x) - \int\limits_{\Gamma_{M-1}} K_M^{(1,1)}(k,\gamma;x,y)\,u_{M-1}(y)dl(y) - \int\limits_{\Gamma_{M-1}} K_M^{(1,2)}(k,\gamma;x,y)\,v_{M-1}(y)dl(y)$$

$$- \int\limits_{\Gamma_M} K_M^{(1,3)}(k,\gamma;x,y)\,u_M(y)dl(y) - \int\limits_{\Gamma_M} K_M^{(1,4)}(k,\gamma;x,y)\,v_M(y)dl(y) = 0, \quad x \in \Gamma_M, \tag{3}$$

$$v_1(x) - \int\limits_{\Gamma_1} K_1^{(2,3)}(k,\gamma;x,y)\,u_1(y)dl(y) - \int\limits_{\Gamma_1} K_1^{(2,4)}(k,\gamma;x,y)\,v_1(y)dl(y)$$

$$- \int\limits_{\Gamma_2} K_1^{(2,5)}(k,\gamma;x,y)\,u_2(y)dl(y) - \int\limits_{\Gamma_2} K_1^{(2,6)}(k,\gamma;x,y)\,v_2(y)dl(y) = 0, \quad x \in \Gamma_1, \tag{4}$$

$$v_m(x) - \int\limits_{\Gamma_{m-1}} K_m^{(2,1)}(k,\gamma;x,y)\, u_{m-1}(y) dl(y) - \int\limits_{\Gamma_{m-1}} K_m^{(2,2)}(k,\gamma;x,y)\, v_{m-1}(y) dl(y)$$
$$- \int\limits_{\Gamma_m} K_m^{(2,3)}(k,\gamma;x,y)\, u_m(y) dl(y) - \int\limits_{\Gamma_m} K_m^{(2,4)}(k,\gamma;x,y)\, v_m(y) dl(y)$$
$$- \int\limits_{\Gamma_{m+1}} K_m^{(2,5)}(k,\gamma;x,y)\, u_{m+1}(y) dl(y) - \int\limits_{\Gamma_{m+1}} K_m^{(2,6)}(k,\gamma;x,y)\, v_{m+1}(y) dl(y) = 0, \tag{5}$$

where $x \in \Gamma_m$, $m = 2, 3, \dots, M-1$,

$$v_M(x) - \int\limits_{\Gamma_{M-1}} K_M^{(2,1)}(k,\gamma;x,y)\, u_{M-1}(y) dl(y) - \int\limits_{\Gamma_{M-1}} K_M^{(2,2)}(k,\gamma;x,y)\, v_{M-1}(y) dl(y)$$
$$- \int\limits_{\Gamma_M} K_M^{(2,3)}(k,\gamma;x,y)\, u_M(y) dl(y) - \int\limits_{\Gamma_M} K_M^{(2,4)}(k,\gamma;x,y)\, v_M(y) dl(y) = 0, \quad x \in \Gamma_M. \tag{6}$$

Here,

$$K_m^{(1,1)}(x,y) = \frac{\partial G_m(x,y)}{\partial n(y)}, \quad K_m^{(1,2)}(x,y) = -\frac{2\eta_{m-1} G_m(x,y)}{\eta_m + \eta_{m-1}}, \tag{7}$$

$$K_m^{(1,3)}(x,y) = \frac{\partial \left(G_{m+1}(x,y) - G_m(x,y)\right)}{\partial n(y)},$$
$$K_m^{(1,4)}(x,y) = \frac{2\left(\eta_{m+1} G_m(x,y) - \eta_m G_{m+1}(x,y)\right)}{\eta_{m+1} + \eta_m}, \tag{8}$$

$$K_m^{(1,5)}(x,y) = -\frac{\partial G_{m+1}(x,y)}{\partial n(y)}, \quad K_m^{(1,6)}(x,y) = \frac{2\eta_{m+2} G_{m+1}(x,y)}{\eta_{m+2} + \eta_{m+1}}, \tag{9}$$

$$K_m^{(2,1)}(x,y) = \frac{\partial^2 G_m(x,y)}{\partial n(x) \partial n(y)}, \quad K_m^{(2,2)}(x,y) = -\frac{2\eta_{m-1}}{\eta_m + \eta_{m-1}} \frac{\partial G_m(x,y)}{\partial n(x)}, \tag{10}$$

$$K_m^{(2,3)}(x,y) = \frac{\partial^2 \left(G_{m+1}(x,y) - G_m(x,y)\right)}{\partial n(x) \partial n(y)},$$

$$K_m^{(2,4)}(x,y) = \frac{2\eta_{m+1}}{\eta_{m+1} + \eta_m} \frac{\partial G_m(x,y)}{\partial n(x)} - \frac{2\eta_m}{\eta_{m+1} + \eta_m} \frac{\partial G_{m+1}(x,y)}{\partial n(x)}, \tag{11}$$

$$K_m^{(2,5)}(x,y) = -\frac{\partial^2 G_{m+1}(x,y)}{\partial n(x) \partial n(y)}, \quad K_m^{(2,6)}(x,y) = \frac{2\eta_{m+2}}{\eta_{m+2} + \eta_{m+1}} \frac{\partial G_{m+1}(x,y)}{\partial n(x)}, \tag{12}$$

$$G_m(k,\gamma;x,y) = \frac{i}{4} H_0^{(1)}\left(k_m |x-y|\right), \quad m \in \mathbb{M} \cup o, \tag{13}$$

where $\mathbb{M} = \{1, 2, \dots, M\}$, $o = M+1$, and $H_0^{(1)}$ is the Hankel function of the first kind and zero index (see, e.g., [1], p. 360). We assume that each contour Γ_m is 2-times differentiable and closed, and all these contours are disjoint (see Fig. 1). By n we denote the outer normal unit vector to the boundary Γ_m.

The coefficients k_m are equal to $k\nu_m$, where $k \in \mathbb{L}$ is the free-space wavenumber, $\eta_m = \nu_m^{-2}$ for H-polarized field and $\eta_m = 1$ for E-polarization. Here $\mathbb{L}$ is the Riemann surface of the function $\ln k$.

We assume that $\mathbb{E}, \mathbb{A}, \mathbb{P} \subseteq \mathbb{M}$ are sets of indexes such that $\mathbb{E} \neq \emptyset$, $\mathbb{E} \cap \mathbb{A} \cap \mathbb{P} = \emptyset$, $\mathbb{E} \cup \mathbb{A} \cup \mathbb{P} = \mathbb{M}$. In each active region Ω_e, $e \in \mathbb{E}$, the refractive index $\nu_e = \alpha_e - i\gamma$ is complex-valued with positive imaginary part γ named the threshold gain. In each region with absorption Ω_a, $a \in \mathbb{A}$, we write $\nu_a = \alpha_a + i\delta_a$, where $\delta_a > 0$ is the absorption index. In each passive region Ω_p, $p \in \mathbb{P}$, and the unbounded domain $\Omega_o = \mathbb{R}^2 \setminus \bigcup_{m=1}^{M} \overline{\Omega}_m$ the refractive index is equal to real numbers $\nu_p = \alpha_p$ and $\nu_o = \alpha_o$, respectively. All the coefficients α_m are positive.

In LEP for microcavities with active regions [14] we look for $k \in \mathbb{L}$ and $\gamma > 0$ such that there exist nontrivial solutions of system (1)–(6). All other parameters are given. If $M = 1$, then, using equations (1)–(6), we obtain the system of BIEs for the problem for microcavities with uniform gain [13]. If $M = 1$ and $\gamma = 0$, then we get the system of BIEs for classic problem for passive microcavities (see, for example, [2, 3]). Therefore, we investigate the problem and for $\gamma = 0$.

3. WEAK SINGULARITY OF THE KERNELS

Clearly, for $i = 1, 2$ the kernels $K_m^{(i,1)}(x, y)$ and $K_m^{(i,2)}(x, y)$, where $m = 2, 3, \ldots, M$, and the kernels $K_m^{(i,5)}(x, y)$ and $K_m^{(i,6)}(x, y)$, where $m = 1, 2, \ldots, M-1$, are continuous in $x \in \Gamma_l$ and $y \in \Gamma_p$, since $l \neq p$. In this section we prove that the kernels $K_m^{(1,3)}(x, y)$ and $K_m^{(2,4)}(x, y)$, $m \in \mathbb{M}$, where $x, y \in \Gamma_m$, are also continuous. In addition, if $\eta_m = \eta_o$, then the kernels $K_m^{(1,4)}$, $m \in \mathbb{M}$, are continuous, else $K_m^{(1,4)}$ have logarithmic singularities. The kernels $K_m^{(2,3)}$, $m \in \mathbb{M}$, always have logarithmic singularities.

Below we prove the corresponding assertions, but previously we present the following well known statement (see, e.g., [17], p. 384). If a curve Γ_m has a continuous curvature, then

$$\lim_{|x-y|\to 0} \frac{((x-y)\cdot n(x))}{|x-y|^2} = \frac{\xi(x)}{2}, \quad x, y \in \Gamma_m, \tag{14}$$

where ξ is the curvature of the curve Γ_m. Here by "$\cdot$" we denote the standard inner product on $\mathbb{R}^2$.

Lemma 1. *For each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+ = \{\gamma \geq 0\}$ we have*

$$\lim_{|x-y|\to 0} K_m^{(1,3)}(k, \gamma; x, y) = 0, \quad x, y \in \Gamma_m, \quad m \in \mathbb{M}.$$

Proof. Let us recall (see, e.g., [1], p. 360) that

$$H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z), \quad \left(H_0^{(1)}(z)\right)_z' = -H_1^{(1)}(z), \tag{15}$$

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-z^2/4\right)^k}{k!\Gamma(\nu+k+1)}, \quad N_\nu(z) = -\frac{(z/2)^{-\nu}}{\pi} \sum_{k=0}^{\nu-1} \frac{(\nu-k-1)!}{k!} \left(\frac{z^2}{4}\right)^k + \frac{2}{\pi} \ln \frac{z}{2} J_\nu(z)$$
$$- \frac{(z/2)^\nu}{\pi} \sum_{k=0}^{\infty} \frac{\left(\psi(k+1) + \psi(\nu+k+1)\right)\left(-z^2/4\right)^k}{k!(\nu+k)!}, \tag{16}$$

where ν is a positive integer. Here $\Gamma(\nu)$ is the gamma function, $\Gamma(\nu+1) = \nu!$, and $\psi(\nu)$ is the digamma function, $\psi(\nu) = \psi(1) + \sum_{k=0}^{\nu} k^{-1}$, $\nu \geq 2$. Using (16), we get

$$\lim_{z\to 0} zJ_1(z) = \lim_{z\to 0} \frac{z^2}{2} \sum_{k=0}^{\infty} \frac{\left(-z^2/4\right)^k}{k!(k+1)!} = 0,$$

$$\lim_{z\to 0} zN_1(z) = \lim_{z\to 0} z \left(-\frac{(z/2)^{-1}}{\pi} + \frac{2}{\pi} \ln \frac{z}{2} J_1(z) - \frac{z/2}{\pi} \sum_{k=0}^{\infty} \frac{\left(\psi(k+1) + \psi(k+2)\right)\left(-z^2/4\right)^k}{k!(k+1)!} \right) = -\frac{2}{\pi}.$$

It follows from the two previous equalities and (15) that

$$\lim_{z\to 0} zH_1^{(1)}(z) = \lim_{z\to 0} zJ_1(z) + i\lim_{z\to 0} zN_1(z) = -\frac{2i}{\pi}. \tag{17}$$

By definition,

$$\frac{\partial u(x)}{\partial n(x)} = (\mathrm{grad}_x u \cdot n(x)), \quad \frac{\partial |x-y|}{\partial n(y)} = \left(\mathrm{grad}_y |x-y| \cdot n(y)\right) = -\frac{((x-y)\cdot n(y))}{|x-y|}, \tag{18}$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Combining (13), (15), and (18), we see that

$$\frac{\partial G_p(x,y)}{\partial n(y)} = \frac{i}{4} k_p H_1^{(1)}(k_p|x-y|)\frac{((x-y)\cdot n(y))}{|x-y|} = \frac{i}{4} k_p |x-y| H_1^{(1)}(k_p|x-y|)\frac{((x-y)\cdot n(y))}{|x-y|^2}, \tag{19}$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Here and below $p = m,\ m+1$, and if $m = M$, then $m+1 = o$. Therefore, using (19), (14), and (17), we obtain

$$\lim_{|x-y|\to 0} \frac{\partial G_p(x,y)}{\partial n(y)} = \frac{i}{4}\left(-\frac{2i}{\pi}\right)\left(-\frac{\xi(y)}{2}\right) = -\frac{\xi(y)}{4\pi}, \tag{20}$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Thus, combining (8) and (20), for each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$, we finally get

$$\lim_{|x-y|\to 0} K_m^{(1,3)}(k,\gamma;x,y) = \lim_{|x-y|\to 0}\left(\frac{\partial G_{m+1}(k;x,y)}{\partial n(y)} - \frac{\partial G_m(k,\gamma;x,y)}{\partial n(y)}\right) = -\frac{\xi(y)}{4\pi} + \frac{\xi(y)}{4\pi} = 0,$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. □

Lemma 2. *For each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ we have*

$$\lim_{|x-y|\to 0} K_m^{(2,4)}(k,\gamma;x,y) = \frac{\xi(x)}{2\pi}\left(\frac{\eta_m - \eta_{m+1}}{\eta_{m+1} + \eta_m}\right), \quad x, y \in \Gamma_m, \quad m \in \mathbb{M}.$$

Proof. The proof is analogous to the proof of Lemma 1. Indeed, arguing as in (18), we have

$$\frac{\partial |x-y|}{\partial n(x)} = (\mathrm{grad}_x |x-y| \cdot n(x)) = \frac{((x-y)\cdot n(x))}{|x-y|}, \tag{21}$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Combining (13), (18), (21), and (15), we see that

$$\frac{\partial G_p(x,y)}{\partial n(x)} = -\frac{i}{4} k_p H_1^{(1)}(k_p|x-y|)\frac{((x-y)\cdot n(x))}{|x-y|} = -\frac{i}{4} k_p |x-y| H_1^{(1)}(k_p|x-y|)\frac{((x-y)\cdot n(x))}{|x-y|^2},$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Here and below $p = m,\ m+1$, and if $m = M$, then $m+1 = o$. Therefore, using the last equality, (14), and (17), we obtain

$$\lim_{|x-y|\to 0} \frac{\partial G_p(x,y)}{\partial n(x)} = -\frac{i}{4}\left(-\frac{2i}{\pi}\right)\frac{\xi(x)}{2} = -\frac{\xi(x)}{4\pi}, \tag{22}$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Thus, combining (11) and (22), for each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ we finally get

$$\lim_{|x-y|\to 0} K_m^{(2,4)}(k,\gamma;x,y) = \lim_{|x-y|\to 0}\left(\frac{2\eta_{m+1}}{\eta_{m+1}+\eta_m}\frac{\partial G_m(k,\gamma;x,y)}{\partial n(x)} - \frac{2\eta_m}{\eta_{m+1}+\eta_m}\frac{\partial G_{m+1}(k;x,y)}{\partial n(x)}\right)$$

$$= -\frac{2\eta_{m+1}}{\eta_{m+1}+\eta_m}\frac{\xi(x)}{4\pi} + \frac{2\eta_m}{\eta_{m+1}+\eta_m}\frac{\xi(x)}{4\pi} = \frac{\xi(x)}{2\pi}\left(\frac{\eta_m - \eta_{m+1}}{\eta_{m+1}+\eta_m}\right), \quad x, y \in \Gamma_m, \quad m \in \mathbb{M},$$

as desired. □

Lemma 3. *For each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ we have*

$$\lim_{|x-y|\to 0} \frac{K_m^{(1,4)}(k,\gamma;x,y)}{\ln|x-y|} = \frac{(\eta_m - \eta_{m+1})}{\pi(\eta_{m+1}+\eta_m)},$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$ (if $m = M$, then $m+1 = o$).

Proof. It is well known that (see, e.g., [1], p. 360)

$$\lim_{z\to 0} J_0(z) = 1. \tag{23}$$

It follows from (16) that

$$N_0(z) = \frac{2}{\pi} J_0(z) \ln(z/2) - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k} \psi(k+1)}{(k!)^2}.$$

Therefore,

$$\lim_{z \to 0} \frac{N_0(z)}{\ln z} = \frac{2}{\pi}. \tag{24}$$

Combining now (15), (23) and (24), we see that

$$\lim_{z \to 0} \frac{H_0^{(1)}(z)}{\ln z} = \frac{2i}{\pi}. \tag{25}$$

Using (8) and (13), we obtain

$$\begin{aligned}
\frac{K_m^{(1,4)}(k,\gamma;x,y)}{\ln|x-y|} &= \frac{2\eta_{m+1}}{\eta_{m+1}+\eta_m} \frac{G_m(x,y)}{\ln|x-y|} - \frac{2\eta_m}{\eta_{m+1}+\eta_m} \frac{G_{m+1}(x,y)}{\ln|x-y|} \\
&= \frac{i}{4} \frac{2\eta_{m+1}}{\eta_{m+1}+\eta_m} \frac{H_0^{(1)}(k_m|x-y|)}{\ln|x-y|} - \frac{i}{4} \frac{2\eta_m}{\eta_{m+1}+\eta_m} \frac{H_0^{(1)}(k_{m+1}|x-y|)}{\ln|x-y|} \\
&= \frac{i}{2} \frac{\eta_{m+1}}{\eta_{m+1}+\eta_m} \frac{H_0^{(1)}(k_m|x-y|)}{\ln(k_m|x-y|)} \frac{\ln|x-y| + \ln k_m}{\ln|x-y|} \\
&\quad - \frac{i}{2} \frac{\eta_m}{\eta_{m+1}+\eta_m} \frac{H_0^{(1)}(k_{m+1}|x-y|)}{\ln(k_{m+1}|x-y|)} \frac{\ln|x-y| + \ln k_{m+1}}{\ln|x-y|},
\end{aligned} \tag{26}$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Thus, using (25) and (26), for each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ we obtain

$$\lim_{|x-y| \to 0} \frac{K_m^{(1,4)}(k,\gamma;x,y)}{\ln|x-y|} = \frac{i}{2} \frac{\eta_{m+1}}{\eta_{m+1}+\eta_m} \frac{2i}{\pi} - \frac{i}{2} \frac{\eta_m}{\eta_{m+1}+\eta_m} \frac{2i}{\pi} = \frac{\eta_m - \eta_{m+1}}{\pi(\eta_{m+1}+\eta_m)},$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$ (if $m = M$, then $m+1 = o$). □

Lemma 4. *For each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ we have*

$$\lim_{|x-y| \to 0} \frac{K_m^{(2,3)}(k,\gamma;x,y)}{\ln|x-y|} = \frac{k_m^2 - k_{m+1}^2}{4\pi},$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$ (if $m = M$, then $m+1 = o$).

Proof. Clearly,

$$\frac{\partial\left((x-y)\cdot n(y)\right)}{\partial n(x)} = \left(\frac{\partial(x-y)}{\partial n(x)} \cdot n(y)\right) = (n(x)\cdot n(y)), \tag{27}$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Let us recall (see, e.g., [1], p. 361) that

$$\left(H_1^{(1)}(z)\right)_z' = -H_2^{(1)}(z) + \frac{1}{z} H_1^{(1)}(z). \tag{28}$$

Combining (27) and (21), we get

$$\begin{aligned}
\frac{\partial}{\partial n(x)} \left(\frac{((x-y)\cdot n(y))}{|x-y|}\right) &= \left(\frac{\partial\left((x-y)\cdot n(y)\right)}{\partial n(x)} |x-y| - ((x-y)\cdot n(y)) \frac{\partial |x-y|}{\partial n(x)}\right) |x-y|^{-2} \\
&= \frac{(n(x)\cdot n(y))}{|x-y|} - \frac{((x-y)\cdot n(x))\,((x-y)\cdot n(y))}{|x-y|^3}, \quad x, y \in \Gamma_m.
\end{aligned} \tag{29}$$

Combining now (19), (28), and (29), we obtain

$$\frac{\partial^2 G_p(x,y)}{\partial n(x)\partial n(y)} = \frac{\partial}{\partial n(x)} \left(\frac{\partial G_p(x,y)}{\partial n(y)}\right) = \frac{\partial}{\partial n(x)} \left(\frac{i}{4} k_p H_1^{(1)}(k_p|x-y|) \frac{((x-y)\cdot n(y))}{|x-y|}\right)$$

$$= -\frac{ik_p^2}{4} H_2^{(1)}(k_p|x-y|) \frac{((x-y)\cdot n(y))((x-y)\cdot n(x))}{|x-y|^2} + \frac{ik_p}{4} H_1^{(1)}(k_p|x-y|) \frac{(n(x)\cdot n(y))}{|x-y|}, \quad (30)$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$. Here and below $p = m, m+1$, and if $m = M$, then $m+1 = o$. We denote

$$C_p(x,y) = k_p^2 H_2^{(1)}(k_p|x-y|) \frac{((x-y)\cdot n(y))((x-y)\cdot n(x))}{|x-y|^2}, \quad x, y \in \Gamma_m, \quad m \in \mathbb{M}, \quad (31)$$

$$D_p(x,y) = k_p H_1^{(1)}(k_p|x-y|) \frac{(n(x)\cdot n(y))}{|x-y|}, \quad x, y \in \Gamma_m, \quad m \in \mathbb{M}. \quad (32)$$

Then we can rewrite (30) in the form

$$\frac{\partial^2 G_p(x,y)}{\partial n(x)\partial n(y)} = -\frac{i}{4}\left(C_p(x,y) - D_p(x,y)\right), \quad x, y \in \Gamma_m, \quad m \in \mathbb{M}. \quad (33)$$

Using (16), we see that

$$\lim_{z\to 0} z^2 J_2(z) = \lim_{z\to 0}\left(\frac{z^4}{4}\sum_{k=0}^{\infty}\frac{(-z^2/4)^k}{k!(k+2)!}\right) = 0. \quad (34)$$

Now using (16) and (34), we get

$$\lim_{z\to 0}\frac{z^2 N_2(z)}{\ln z} = \lim_{z\to 0}\frac{z^2}{\ln z}\left((-1)\frac{(z/2)^{-2}}{\pi}\left(1+\frac{z^2}{4}\right) + \frac{2}{\pi}\ln\frac{z}{2} J_2(z)\right.$$
$$\left. - \frac{(z/2)^2}{\pi}\sum_{k=0}^{\infty}\frac{\left(\psi(k+1)+\psi(k+3)\right)(-z^2/4)^k}{k!(k+2)!}\right) = 0. \quad (35)$$

It follows from (34), (35), and (15) that

$$\lim_{z\to 0}\frac{z^2 H_2^{(1)}(z)}{\ln z} = \lim_{z\to 0}\frac{z^2 J_2(z)}{\ln z} + i\lim_{z\to 0}\frac{z^2 N_2(z)}{\ln z} = 0. \quad (36)$$

It follows from (31) that

$$\frac{C_p(x,y)}{\ln|x-y|} = \frac{k_p^2 H_2^{(1)}(k_p|x-y|)}{\ln|x-y|}\frac{((x-y)\cdot n(y))((x-y)\cdot n(x))}{|x-y|^2} = \frac{((x-y)\cdot n(x))}{|x-y|^2}$$
$$\times \frac{((x-y)\cdot n(y))}{|x-y|^2}\frac{k_p^2|x-y|^2 H_2^{(1)}(k_p|x-y|)}{\ln(k_p|x-y|)}\frac{\ln|x-y| + \ln k_p}{\ln|x-y|}, \quad x, y \in \Gamma_m, \quad m \in \mathbb{M}. \quad (37)$$

Now using (37), (36), and (14), we obtain

$$\lim_{|x-y|\to 0}\left(\frac{C_{m+1}(x,y) - C_m(x,y)}{\ln|x-y|}\right) = 0, \quad (38)$$

where $x, y \in \Gamma_m$, $m \in \mathbb{M}$ (if $m = M$, then $m+1 = o$). Using (16), we get

$$\frac{1}{z}N_1(z) = \frac{1}{z}\left(-\frac{(z/2)^{-1}}{\pi} + \frac{2}{\pi}\ln\frac{z}{2}J_1(z) - \frac{(z/2)}{\pi}\sum_{k=0}^{\infty}\frac{\left(\psi(k+1)+\psi(k+2)\right)(-z^2/4)^k}{k!(k+1)!}\right)$$
$$= \left(-\frac{2}{\pi z^2} + \frac{2}{\pi z}J_1(z)\ln\frac{z}{2} - f(z)\right), \quad (39)$$

where

$$f(z) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{\left(\psi(k+1)+\psi(k+2)\right)(-z^2/4)^k}{k!(k+1)!}. \quad (40)$$

Let us recall (see, e.g., [1], p. 258) that $\psi(z+1)=\psi(z)+1/z$. Therefore, taking the limit in (40), we get

$$\lim_{z\to 0} f(z)=\frac{1}{2\pi}\left(\psi(1)+\psi(2)\right)=\frac{1}{2\pi}\left(2\psi(1)+1\right)=\frac{1}{\pi}\left(\psi(1)+\frac{1}{2}\right). \quad (41)$$

It follows from (15) and (39) that

$$\frac{1}{z}H_1^{(1)}(z)=\frac{1}{z}J_1(z)+i\frac{1}{z}N_2(z)=\frac{1}{z}J_1(z)+i\left(-\frac{2}{\pi z^2}+\frac{2}{\pi z}J_1(z)\ln\frac{z}{2}-f(z)\right). \quad (42)$$

Using (32) and (42), we obtain

$$D_p(x,y)=k_pH_1^{(1)}(k_p|x-y|)\frac{(n(x)\cdot n(y))}{|x-y|}=k_p^2(n(x)\cdot n(y))\frac{J_1(k_p|x-y|)}{k_p|x-y|}-\frac{2i(n(x)\cdot n(y))}{\pi|x-y|^2}$$
$$+\frac{2k_p^2 i(n(x)\cdot n(y))}{\pi}\frac{J_1(k_p|x-y|)}{k_p|x-y|}\ln\frac{k_p|x-y|}{2}-ik_p^2(n(x)\cdot n(y))f(k_p|x-y|),$$
$$x,y\in\Gamma_m,\quad m\in\mathbb{M}.$$

Therefore,

$$\frac{D_{m+1}(x,y)-D_m(x,y)}{\ln|x-y|}=\frac{k_{m+1}^2(n(x)\cdot n(y))}{\ln|x-y|}\frac{J_1(k_{m+1}|x-y|)}{k_{m+1}|x-y|}-\frac{k_m^2(n(x)\cdot n(y))}{\ln|x-y|}\frac{J_1(k_m|x-y|)}{k_m|x-y|}$$
$$+\frac{2k_{m+1}^2 i(n(x)\cdot n(y))\left(\ln|x-y|+\ln k_{m+1}-\ln 2\right)}{\pi\ln|x-y|}\frac{J_1(k_{m+1}|x-y|)}{k_{m+1}|x-y|}$$
$$-\frac{2k_m^2 i(n(x)\cdot n(y))\left(\ln|x-y|+\ln k_m-\ln 2\right)}{\pi\ln|x-y|}\frac{J_1(k_m|x-y|)}{k_m|x-y|}$$
$$-\frac{ik_{m+1}^2(n(x)\cdot n(y))f(k_{m+1}|x-y|)+ik_m^2(n(x)\cdot n(y))f(k_m|x-y|)}{\ln|x-y|}, \quad (43)$$

where $x,y\in\Gamma_m$, $m\in\mathbb{M}$ (if $m=M$, then $m+1=o$). Using (16), we get

$$\lim_{z\to 0}\frac{1}{z}J_1(z)=\lim_{z\to 0}\frac{1}{z}\frac{z}{2}\sum_{k=0}^{\infty}\frac{\left(-z^2/4\right)^k}{k!(k+1)!}=\lim_{z\to 0}\left(\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\left(-z^2/4\right)^k}{k!(k+1)!}\right)=\frac{1}{2}. \quad (44)$$

It follows from (33) and (43) that

$$\frac{K_m^{(2,3)}(k,\gamma;x,y)}{\ln|x-y|}=-\frac{i}{4}\left(\frac{C_{m+1}(x,y)-C_m(x,y)}{\ln|x-y|}\right)+\frac{i}{4}\left(\frac{D_{m+1}(x,y)-D_m(x,y)}{\ln|x-y|}\right), \quad (45)$$

where $x,y\in\Gamma_m$, $m\in\mathbb{M}$ (if $m=M$, then $m+1=o$). Thus, combining now (45), (38), (44), and (41), we obtain the desired assertion. □

4. SPECTRAL PROBLEM FOR THE FREDHOLM OPERATOR FUNCTION

By $C(\Gamma_m)$ we denote the Banach space of continuous on Γ_m, $m\in\mathbb{M}$, functions with the usual maximum norm (see, e.g., [9], p. 3)

$$||u||_{m,\infty}=\max_{x\in\Gamma_m}|u(x)|,\quad m\in\mathbb{M}.$$

We introduce the following integral operators with the kernels defined in (7)–(12):

$$\left(B_m^{(i,j)}(k,\gamma)w_s^{(j)}\right)(x)=\int_{\Gamma_s}K_m^{(i,j)}(k,\gamma;x,y)w_s^{(j)}(y)dl(y),\quad x\in\Gamma_m,$$

where $k\in\mathbb{L}$ and $\gamma\in\mathbb{R}_+$, $i=1,2$,

$$s=\begin{cases}m-1, & \text{for } j=1,2, \quad m=2,3,\ldots,M,\\ m, & \text{for } j=3,4, \quad m=1,2,\ldots,M,\\ m+1, & \text{for } j=5,6, \quad m=1,2,\ldots,M-1,\end{cases}\qquad w_s^{(j)}=\begin{cases}u_s, & \text{for } j=1,3,5,\\ v_s, & \text{for } j=2,4,6.\end{cases}$$

As we have seen in the previous section, for $i = 1, 2$ the kernels $K_m^{(i,1)}$, $K_m^{(i,2)}$, where $m = 2, 3, \ldots, M$, $K_m^{(i,5)}$, $K_m^{(i,6)}$, where $m = 1, 2, \ldots, M-1$, and $K_m^{(1,3)}(x,y)$, $K_m^{(2,4)}(x,y)$, where $m \in \mathbb{M}$, are continuous. If $\eta_m = \eta_o$, then the kernels $K_m^{(1,4)}$, $m \in \mathbb{M}$, are continuous, else $K_m^{(1,4)}$ are weakly singular. The kernels $K_m^{(2,3)}$, $m \in \mathbb{M}$, are weakly singular. Therefore (see, e.g., Theorem 2.8, p. 17, and Problem 2.3, p. 27, [9]) for each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ the operators $B_m^{(i,j)} : C(\Gamma_s) \to C(\Gamma_m)$, where $j = 1, 2$, $i = 3, 4$, are bounded with

$$||B_m^{(i,j)}(k,\gamma)||_\infty = \max_{x \in \Gamma_m} \int\limits_{\Gamma_s} |K_m^{(i,j)}(k,\gamma; x, y)| dl(y) < \infty. \tag{46}$$

Moreover, these integral operators are compact (see, e.g., Theorem 2.23, p. 26, [9]). We can prove analogously that for $s = m-1$, $j = 1, 2$ and $s = m+1$, $j = 5, 6$, these operators are bounded (the upper bound of form (46) holds true) and compact.

For each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ we introduce the operator $\mathbf{B} : W \to W$, $W = W_1 \times W_2 \times \ldots \times W_M$, $W_m = C(\Gamma_m) \times C(\Gamma_m)$, $m \in \mathbb{M}$,

$$\mathbf{B}(k,\gamma) = \begin{pmatrix} B_1(k,\gamma) & 0 & 0 & \ldots & 0 \\ B_2(k,\gamma) & 0 & 0 & \ddots & \vdots \\ 0 & B_3(k,\gamma) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \ldots & 0 & B_{M-2}(k,\gamma) & 0 \\ 0 & \ldots & 0 & 0 & B_{M-1}(k,\gamma) \\ 0 & \ldots & 0 & 0 & B_M(k,\gamma) \end{pmatrix}.$$

Here,

$$B_1 = \begin{pmatrix} B_1^{(1,3)} & B_1^{(1,4)} & B_1^{(1,5)} & B_1^{(1,6)} \\ B_1^{(2,3)} & B_1^{(2,4)} & B_1^{(2,5)} & B_1^{(2,6)} \end{pmatrix}, \quad B_M = \begin{pmatrix} B_M^{(1,1)} & B_M^{(1,2)} & B_M^{(1,3)} & B_M^{(1,4)} \\ B_M^{(2,1)} & B_M^{(2,2)} & B_M^{(2,3)} & B_M^{(2,4)} \end{pmatrix},$$

$$B_m = \begin{pmatrix} B_m^{(1,1)} & B_m^{(1,2)} & B_m^{(1,3)} & B_m^{(1,4)} & B_m^{(1,5)} & B_m^{(1,6)} \\ B_m^{(2,1)} & B_m^{(2,2)} & B_m^{(2,3)} & B_m^{(2,4)} & B_m^{(2,5)} & B_m^{(2,6)} \end{pmatrix}, \quad m = 2, 3, \ldots, M-1.$$

All the introduced operators are compact. Thus, the following theorem is true.

Theorem 1. *For each $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ the integral operator $\mathbf{B} : W \to W$ is compact.*

Now we can rewrite system (1)–(6) in the form

$$\mathbf{w} = \mathbf{B}(k,\gamma)\mathbf{w} \tag{47}$$

and look for $k \in \mathbb{L}$ and $\gamma \in \mathbb{R}_+$ such that there exist nontrivial solutions $\mathbf{w} \in W$ of operator equation (47). We finally note that the operator $I - \mathbf{B}$ is Fredholm with index zero. Here I is the identical operator in the space W.

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