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LASING EFFECT FROM THE VIEWPOINT OF MATHEMATICAL THEORY OF EIGENVALUE PROBLEMS FOR DIELECTRIC OPEN RESONATORS

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From the viewpoint of mathematical modeling based on the so-called ***cold models*** of lasers, the lasing is the existence of at least one real-valued natural frequency of an inhomogeneously filled dielectric open resonator. This may happen provided that there is a sub-domain, inside the resonator, that is occupied with an ***active material*** characterized by positive ***gain*** (equivalently, negative loss factor or negative imaginary part of dielectric permittivity). Therefore we consider several generalized eigenvalue problems for the Helmholtz and Maxwell equations, where the fields depend on time as e^{-ikct} and the normalized frequency k is eigenparameter: (i) localized 2D dielectric resonator in a homogeneous medium, (ii) in a PEC-wall waveguide, and (iii) in a stratified dielectric medium, (iv) localized 3D dielectric resonator in a homogeneous medium, (v) in a PEC-wall waveguide or a stratified medium, (vi) and near a regular or periodic fiber.

These problems are about the time-harmonic electromagnetic fields in and out of bounded penetrable objects placed in unbounded host medium. Their correct statement needs certain condition imposed on the field behavior at infinity in the 2-D or 3-D background space (also called host medium). Such a condition, in each case, follows from the behavior of the corresponding Green's function analytically continued from the real values of k to the complex domain. Here, the real-k Green's function satisfies, each time, the Principle of Radiation in the form of corresponding ***radiation condition***. Thus, for the real-k counterparts of 2D problems mentioned above, in (i) this is the Sommerfeld condition [1], in (ii) this is the Sveshnikov condition [2], and in (iii) this is the condition established in [3]. Different arrangement of "infinity", in each 2D case, entails different shape of the Green's function. As a result, the domains of their analytic continuation to complex k are also different. In (i) this is the Riemann surface (RS) of $Ln\ k$, in (ii) this is RS of $\Sigma_{n=0}^{\infty}(k^2-(\pi n/d)^2)^{1/2}$, where d is the waveguide-wall separation, and in (iii) this is RS of $Lnk+\Sigma_{n=0}^{\infty}(k^2-k_n^2)^{1/2}$, where k_n are the real-valued critical frequencies of the guided-wave modes of the stratified medium - if they exist. The complex-k conditions at infinity that inherit all the features of real-k radiation conditions and reduce to them at the real axis of the principal sheet of corresponding RS, are called Reichard's conditions [4]. In each case, as one can show by using the Poynting theorem, if dielectric permittivity has zero or positive imaginary part (i.e., the object is *passive*), then the k-eigenvalues can be only complex-valued and have negative imaginary parts on the principal sheet of respective RS. The corresponding eigenfunctions, or modal fields, are destined to decay in time but grow up in space away from the resonator. Therefore these problems are the ***generalized eigenvalue problems***, to distinguish them from classical eigenvalue problems. For the 3D problems, the situation is similar. Modifications are the result of the vector character of 3D electromagnetic fields. For example, instead of Sommerfeld's condition, the real-k case of (iv) needs so-called Silver-Muller radiation condition that eliminates radial components far from the scatterer [5,6]. In the problems (v) and (vi), the real-k radiation conditions are vector analogies of the 2D ones; for (vi) it was established in [3]. One important difference is that the modes guided by a fiber are hybrid and may carry the power in backward direction that must be accounted for in the radiation condition (see [3]). The domain of analytic continuation of the Green's function in k in the case (4) is very simple: this is only a complex plane. However, as soon as the background space contains infinite regular boundaries, e.g. a stratification or a fiber, then this domain turns to a composition of logarithmic sheets of the $Ln(k^2-k_n^2)$ type, with the branch points of RS located at the critical frequencies of the guided modes, k_n.

Therefore, infinite-sheet branching of the domain of continuation of the field function in k is always the price paid for introducing some geometrical infinity. Firstly, neglecting the finite length when switching from the free-space 3-D to the free-space 2-D models leads to the Lnk branching. Secondly, admission of infinite boundaries, either PEC ones or penetrable flat and curved ones, leads to additional $\Sigma_{n=0}^{\infty}(k^2-k_n^2)^{1/2}$ or $\Sigma_{n=0}^{\infty}Ln(k^2-k_n^2)$ branching in the 2-D and 3-D models, respectively. All the eigenvalues, k_s , "live" on a corresponding RS – this is their natural habitat. Based on the theory of integral equations and operator-valued functions [7], one can verify that each of k_s is a piece-wise analytic function of geometry and material parameters. Analyticity can be violated only if two eigenvalues coalesce. They are only of finite multiplicity and may appear or disappear only at branching points and at infinity on RS. In the real-k scattering problems, the field characteristics display the frequency dependences, which are sharply broken at the critical frequencies k_n (at branch points). This is because their derivatives in frequency contain square-root singularities there. An old example of this effect is the Wood's anomalies in the scattering from infinite-periodic diffraction gratings. Here, an elementary cell of the grating can be related to the so-called Floquet virtual waveguide with PEC walls.

Finally, we discuss the effect of lasing. As we have seen, each complex-valued natural frequency k_s of a passive open resonator is analytic function of dielectric permittivity, and hence a continuous function of its imaginary part. If the latter is varied, each natural frequency migrates on the corresponding RS, and if it takes negative values, the frequencies are allowed to migrate across the real axis of the principal sheet of RS into the upper halfplane. Purely real value of every k_s may take place only at a specific value of the gain, γ_s. Therefore, the ***Lasing Eigenvalue Problem*** can be formulated as a homogeneous boundary-value problem for a set of time-harmonic Maxwell (in 3-D) or Helmholtz equations (in 2-D). It must contain an appropriate Silver-Muller (in 3-D) or Sommerfeld (in 2-D) condition at infinity adapted to the type of host medium, and corresponding boundary conditions. It is necessary to find eigenpairs of ***real-valued*** normalized frequency k_s and gain γ_s that generate non-zero solutions, i.e. vector (in 3-D) or scalar (in 2-D) field functions of the lasing modes.

Here, as we have seen above, admission of infinite boundaries in the host medium brings infinite number of branch points located at the real k-axis of the principal sheet of RS. The natural frequencies may appear or disappear only at the boundary of the domain of solution analyticity in k, i.e. at the infinity and at the branch points k_n. Therefore, searching for eigenvalues and tracing their trajectories with the loss/gain variation may potentially hit a situation that k_s coalesces with k_n and then either migrates further on the principal sheet (good) or goes to the other ("non-physical") sheet of RS (bad) or even disappears at all (ugly). Such a "swallowing up by the earth" is of course an exotic thing and tells only about the defect of the model used.

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