# NUMERICAL STUDY OF ELECTROMAGNETIC FIELD SCATTERING BY AN ELECTRICALLY RESISTIVE DISK

**M.V. Balaban [1], R. Sauleau [2], A.I. Nosich [1]**

[1] Institute of Radio-physics and Electronics NASU, Kharkov, Ukraine
[2] IETR, Universite de Rennes 1, Rennes, France
e-mail: mikhail.balaban@gmail.com

***Abstract*** **– In this paper, we study the scattering problem of a given time-harmonic electromagnetic wave by a thin electrically resistive disk. We consider a horizontal ring of electric current as a source of the incident field. To find the solution of the problem, we use analytical-numerical scheme which is based on the generalized boundary conditions, dual integral equations, and the method of analytical regularization. We present the normalized radiation patterns and total radiated power for different values of geometrical and material parameters.**

## I. INTRODUCTION

Electromagnetic wave scattering by thin electrically resistive disk is interesting for many reasons: such a disk is met as a part of printed antennas [1]; it is used as a simplified model of a tree leave [2]. Many approximations and direct computational methods have been used for this problem analysis. However, approximations techniques are commonly used when the size of the disk is much larger then the free-space wavelength and direct computational methods have some problems like large-size matrices to be inverted, low convergence of solution, and hence huge computational time. Here we present the method of spectral domain integral equations (IEs). It is based on the dual IEs formulation [3] and the method of analytical regularization [4].

## II. PROBLEM STATEMENT AND SOLUTION METHOD

Consider the problem of diffraction of the time-harmonic electromagnetic field by an electrically resistive disk of radius $a$ and thickness $\tau$ (Fig. 1). Introduce dimensionless cylindrical coordinates $(\rho = r / a, \varphi, \zeta = z / a)$ with origin in the center of the disk. Assume that the source of the incident field is a ring of electrical current $j_{\varphi}^{e}(r,\varphi,z) = A_0^e \delta(z-d)\delta(r-r_0) / r \cdot \cos m_0 \varphi$. Denote total field as a sum of the scattered and the incident fields: $E = E_{in} + E_{sc}$, $H = H_{in} + H_{sc}$. The scattered field has to satisfy the homogeneous Maxwell equations out of the disk, and the total field has to satisfy the following generalized boundary conditions at the disk median section [5]:

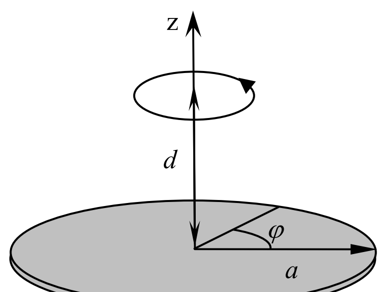



Figure 1. Problem geometry

$$(E_{tg}^{+} - E_{tg}^{-}) = 0\,,\ (E_{tg}^{+} + E_{tg}^{-}) = 2Z_0 R \cdot \vec{n} \times (H_{tg}^{+} - H_{tg}^{-})\,. \tag{1}$$

Here, $Z_0$ is the free-space impedance, and $R$ is the electric resistivity which is given by $R = 1 / Z_0 \sigma \tau$, in case of $k\tau \ll 1$ and $\sigma / \omega \gg 1$, $k = \omega / c$ is the wavenumber, $\sigma$ is the conductivity of the disk material. On the rest part of the disk plane the components of the total field are continuous. Also, the components of the scattered field must satisfy the 3-D radiation condition and the condition of local integrability of power.

Assume the normal to the disk scattered field components are presented in terms of Fourier-Bessel transform as follows:

$$\begin{pmatrix} E_{sc,z}^{\mathrm{sgn}(\zeta)} \\ Z_0 H_{sc,z}^{\mathrm{sgn}(\zeta)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta|} J_m(\kappa\rho) \begin{pmatrix} \kappa e_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \\ \kappa h_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \end{pmatrix} d\kappa\,, \tag{2}$$

Then tangential to the disk scattered field components can be present in terms of the vector Hankel transform [6]:

$$\begin{pmatrix} E_{sc,r}^{\mathrm{sgn}(\zeta)} \\ -iE_{sc,\phi}^{\mathrm{sgn}(\zeta)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta|} \mathbf{H}_m(\kappa\rho) \begin{pmatrix} \mathrm{sgn}(\zeta)\, i\gamma(\kappa) e_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \\ -ka\, h_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \end{pmatrix} d\kappa\,, \tag{3}$$

$$\begin{pmatrix} Z_0 H_{sc,r}^{\mathrm{sgn}(\zeta)} \\ -iZ_0 H_{sc,\phi}^{\mathrm{sgn}(\zeta)} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_0^{\infty} e^{i\gamma(\kappa)|\zeta|} \mathbf{H}_m(\kappa\rho) \begin{pmatrix} \mathrm{sgn}(\zeta)\, i\gamma(\kappa) h_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \\ ka\, e_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa) \end{pmatrix} d\kappa\,, \tag{4}$$

where

$$\mathbf{H}_m(\kappa\rho) = \begin{pmatrix} J'_{|m|}(\kappa\rho) & mJ_{|m|}(\kappa\rho)/(\kappa\rho) \\ mJ_{|m|}(\kappa\rho)/(\kappa\rho) & J'_{|m|}(\kappa\rho) \end{pmatrix} \tag{5}$$

is the matrix kernel of the transform.

Note that the same field expressions can be written for the incident field where one needs to take $e_{m,z}^{in,\mathrm{sgn}(\zeta-\zeta_0)}(\kappa)$, $h_{m,z}^{in,\mathrm{sgn}(\zeta-\zeta_0)}(\kappa)$ instead of $e_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa)$ and $h_{m,z}^{sc,\mathrm{sgn}(\zeta)}(\kappa)$ and $\zeta-\zeta_0$ instead of $\zeta$, $\zeta_0 = d/a$. Also note that thus presented fields components satisfy the radiation condition of Silver-Muller automatically.

Substituting the expressions (2)-(4) into the boundary conditions (1) one can obtain the following set of coupled dual integral equations (IEs) for each *m*-th mode of the Fourier series in terms of $e^{im\phi}$

$$\begin{cases} \int_0^{\infty} \mathbf{H}_m(\kappa\rho) \begin{pmatrix} \gamma(\kappa)(u_m^{sc,-}(\kappa) + u_m^{in,-}(\kappa)) + 2R\,ka\, u_m^{sc,-}(\kappa) \\ ika(v_m^{sc,+}(\kappa) + v_m^{in,+}(\kappa)) + 2Ri\,\gamma(\kappa)\, v_m^{sc,+}(\kappa) \end{pmatrix} d\kappa = \overline{0} & (\rho < 1) \\ \int_0^{\infty} \mathbf{H}_m(\kappa\rho) \begin{pmatrix} ika u_m^{sc,-}(\kappa) \\ -\gamma(\kappa) v_m^{sc,+}(\kappa) \end{pmatrix} d\kappa = \overline{0} & (\rho > 1) \end{cases}, \tag{6}$$

where $u_m^{sc,-}(\kappa) = \left(e_{m,z}^{sc,+}(\kappa) - e_{m,z}^{sc,-}(\kappa)\right)/2$, $u_m^{in,-}(\kappa) = -\mathrm{sgn}(\zeta^{in}) e^{i\gamma(\kappa)|\zeta^{in}|} e_{m,z}^{in,-\mathrm{sgn}(\zeta^{in})}(\kappa)$, $v_m^{sc,+}(\kappa) = \left(h_{m,z}^{sc,+}(\kappa) + h_{m,z}^{sc,-}(\kappa)\right)/2$ and $v_m^{in,+}(\kappa) = e^{i|\zeta^{in}|\gamma(\kappa)} h_{m,z}^{in,-\mathrm{sgn}(\zeta^{in})}(\kappa)$. Also note that $e_{m,z}^{sc,+}(\kappa) + e_{m,z}^{sc,-}(\kappa) = 0$ and $h_{m,z}^{sc,+}(\kappa) - h_{m,z}^{sc,-}(\kappa) = 0$ that follows from the first identity of (1) and the continuity condition of the total field components outside the disk.

Thus our problem is reduced to the set of coupled dual IEs for the unknowns $u_m^{sc,-}(\kappa)$, $v_m^{sc,+}(\kappa)$ wih $u_m^{in,-}(\kappa)$, $v_m^{in,+}(\kappa)$ determined by the incident field functions. In our case (for the co-axial horizontal electrical current ring) they are given in terms of the $e_{m,z}^{in,\mathrm{sgn}(\zeta-\zeta_0)}(\kappa)$ and $h_{m,z}^{in,\mathrm{sgn}(\zeta-\zeta_0)}(\kappa)$ functions by the following expressions $(n = \pm m_0)$:

$$\mathrm{sgn}(n) e_{n,z}^{in,+}(\kappa) = -\mathrm{sgn}(n) e_{n,z}^{in,-}(\kappa) = \frac{iA_0^e \kappa}{4ka}\left(J_{|n|-1}(\kappa\rho_0) + J_{|n|+1}(\kappa\rho_0)\right) \tag{7}$$

$$h_{n,z}^{in,+}(\kappa) = h_{n,z}^{in,-}(\kappa) = \frac{A_0^e \kappa}{4\gamma(\kappa)}\left(J_{|n|-1}(\kappa\rho_0) - J_{|n|+1}(\kappa\rho_0)\right) \tag{8}$$

Here, $A_0^e$ is the given current amplitude, $\rho_0 = r_0/a$ is the normalized radius of the ring, and $m_0$ is the index which corresponds to the azimuth variation of the current.

To find the solutions of (6) we follow the analytical-numerical scheme as [6]

1. Integrate the set of coupled dual IEs and "decouple" them by introducing two unknowns are constants of integration. Obtain two sets of dual IEs.
2. Apply the scholar Hankel integral transform and Titchmarsh-type formula to invert the most singular part of the integral operators of the dual IEs. Obtain Fredholm second kind IEs on the (0,∞) interval.
3. Satisfy the edge condition (or the condition of local integrability of power) to find the additional

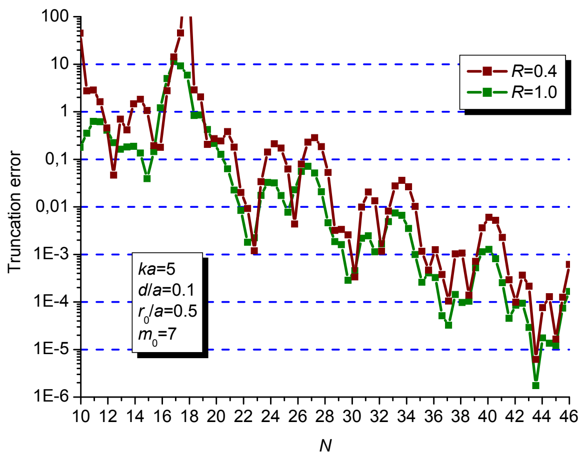



Figure 2. Truncation error vs. truncation number *N*

equations for the constants of integration.

4. Introduce the truncation number $N \geq ka + 1$ and truncate of the interval integration to $(0,N)$.
5. Apply the Nystrom method with the Gauss-type higher-order quadratures to discretize IEs on the $(0,N)$ interval. Find the unknowns at the grid points by inverting the matrix analog of IEs .
6. Find the unknown functions on the $(0,\infty)$ interval by substitution of the set of found values into the Fredolm second kind IEs.

Fig. 2 shows the dependences of the truncation error on the truncation number. One can see a typical behavior of this error which corresponds to the features of the Fredholm second kind IEs.

## III. Numerical results and discusion

Some of preliminary results of computations are presented in Figs. 3 to 6. Fig. 3 shows the dependences of the total power radiated by the horizontal electrical dipole (HED) in the presence of perfectly electrical conductive disk and "*m*-order ring current" (here, modeled by a vanishing-radius ring current with the azimuth variation $m_0 = 1$) in the presence of low-resistivity disk with $R$=0.01. Fig. 4 shows plots of the total radiated power of the same ring current with different azimuth variations ($m_0 = 1, 2, 5, 7$) in the presence of the same low resistivity disk. The first maxima of this quantity on each curve are marked by the numbers 1, 2, 3 and 4, respectively. The incident and total field radiation patterns at these points are presented in Figs. 5.1 – 5.4, respectively. Finally, Fig. 5 shows the dependences of the total radiated power for the ring current source with the azimuth variation $m_0 = 7$ in the presence of the disk of different materials.

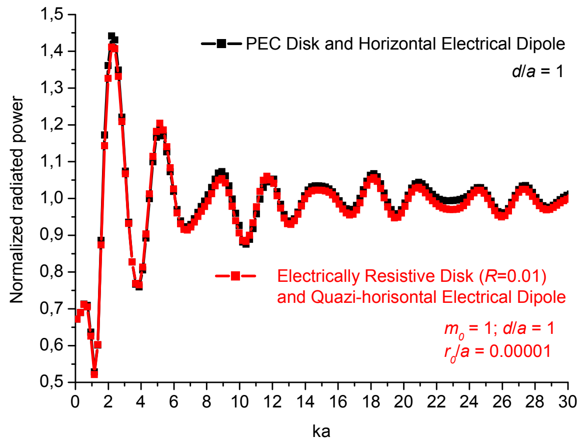



Figure 3. Normalized radiated power as a function of the normalized radius of the disk *ka*

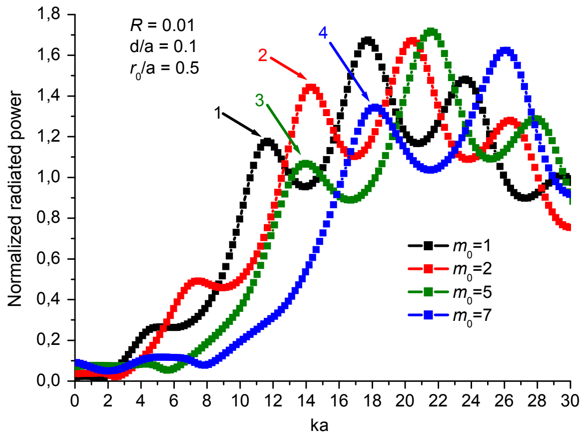
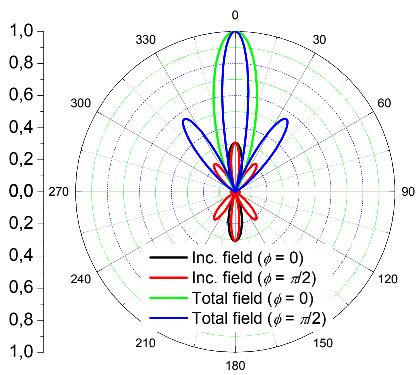
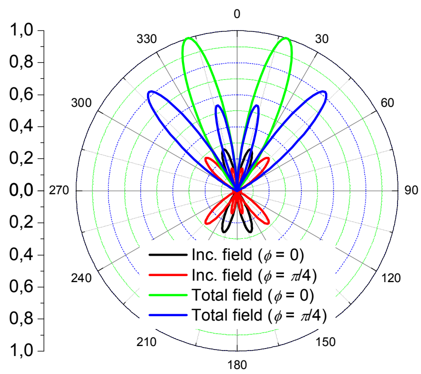



Figure 4. Normalized radiated power as a function of the normalized radius of the disk *ka* for the disk materials $R = 0.01$ and different incident fields




(1)




(2)