5

Simulation and Analysis of Transient Processes in Open Axially-symmetrical Structures: Method of Exact Absorbing Boundary Conditions

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1. Introduction

Present day methodologies for mathematical simulation and computational experiment are generally implemented in electromagnetics through the solution of boundary-value (frequency domain) problems and initial boundary-value (time domain) problems for Maxwell's equations. Most of the results of this theory concerning open resonators have been obtained by the frequency-domain methods. At the same time, a rich variety of applied problems (analysis of complex electrodynamic structures for the devices of vacuum and solid-state electronics, model synthesis of open dispersive structures for resonant quasi-optics, antenna engineering, and high-power electronics, etc.) can be efficiently solved with the help of more universal time-domain algorithms.

The fact that frequency domain approaches are somewhat limited in such problems is the motivation for this study. Moreover, presently known remedies to the various theoretical difficulties in the theory of non-stationary electromagnetic fields are not always satisfactory for practitioners. Such remedies affect the quality of some model problems and limit the capability of time-domain methods for studying transient and stationary processes. One such difficulty is the appropriate and efficient truncation of the computational domain in so-called open problems, i.e. problems where the computational domain is infinite along one or more spatial coordinates. Also, a number of questions occur when solving far-field problems, and problems involving extended sources or sources located in the far-zone.

In the present work, we address these difficulties for the case of TE_{0n} - and TM_{0n} -waves in axially-symmetrical open compact resonators with waveguide feed lines. Sections 2 and 3 are devoted to problem definition. In Sections 4 and 5, we derive exact absorbing conditions for outgoing pulsed waves that enable the replacement of an open problem with an equivalent closed one. In Section 6, we obtain the analytical representation for operators that link the near- and far-field impulsive fields for compact axially-symmetrical structures and consider solutions that allow the use of extended or distant sources. In Section 7, we place some accessory results required for numerical implementation of the approach under consideration. All analytical results are presented in a form that is suitable for using in the finite-difference method on a finite-sized grid and thus is amenable for software implementation. We develop here the approach initiated in the works by Maikov et al. (1986) and Sirenko et al. (2007) and based on the construction of the exact conditions allowing one to reduce an open problem to an equivalent closed one with a bounded domain of analysis. The derived closed problem can then be solved numerically using the standard finite-difference method (Taflove & Hagness, 2000).

In contrast to other well-known approximate methods involving truncation of the computational domain (using, for example, Absorbing Boundary Conditions or Perfectly Matched Layers), our constructed solution is exact, and may be computationally implemented in a way that avoids the problem of unpredictable behavior of computational errors for large observation times. The impact of this approach is most significant in cases of resonant wave scattering, where it results in reliable numerical data.

2. Formulation of the initial boundary-value problem

In Fig. 1, the cross-section of a model for an open axially-symmetrical ($\partial/\partial\phi \equiv 0$) resonant structure is shown, where $\{\rho,\phi,z\}$ are cylindrical and $\{\rho,\vartheta,\phi\}$ are spherical coordinates. By $\Sigma = \Sigma_\phi \times [0,2\pi]$ we denote perfectly conducting surfaces obtained by rotating the curve Σ_ϕ about the z-axis; $\Sigma^{\varepsilon,\sigma} = \Sigma_\phi^{\varepsilon,\sigma} \times [0,2\pi]$ is a similarly defined surface across which the relative

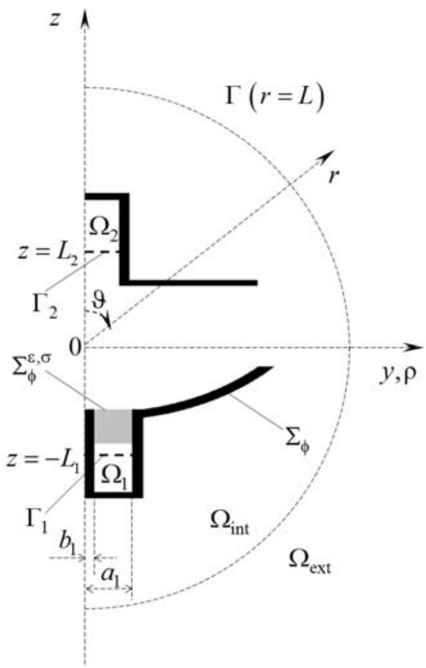

Fig. 1. Geometry of the problem in the half-plane $\phi = \pi/2$.

permittivity $\varepsilon(g)$ and specific conductivity $\sigma_0(g)=\eta_0^{-1}\sigma(g)$ change step-wise; these quantities are piecewise constant inside Ω_{int} and take free space values outside. Here, $g=\{\rho,z\}$; $\eta_0=(\mu_0/\varepsilon_0)^{1/2}$ is the impedance of free space; ε_0, and μ_0 are the electric and magnetic constants of vacuum.

The two-dimensional initial boundary-value problem describing the pulsed axially-symmetrical TE_{0n}- ($E_\rho=E_z=H_\phi\equiv 0$) and TM_{0n}- ($H_\rho=H_z=E_\phi\equiv 0$) wave distribution in open structures of this kind is given by

$$\begin{cases} \left[-\varepsilon(g)\dfrac{\partial^2}{\partial t^2}-\sigma(g)\dfrac{\partial}{\partial t}+\dfrac{\partial^2}{\partial z^2}+\dfrac{\partial}{\partial\rho}\left(\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\rho\right)\right]U(g,t)=F(g,t), \quad t>0, \quad g\in\Omega \\ U(g,t)\big|_{t=0}=\varphi(g), \quad \dfrac{\partial}{\partial t}U(g,t)\Big|_{t=0}=\psi(g), \quad g=\{\rho,z\}\in\bar{\Omega} \\ E_{tg}(p,t)\big|_{p=\{\rho,\phi,z\}\in\Sigma}=0, \quad t\geq 0 \\ E_{tg}(p,t) \text{ and } H_{tg}(p,t) \text{ are continuous when crossing } \Sigma^{\varepsilon,\sigma} \\ U(0,z,t)=0, \quad |z|<\infty, \quad t\geq 0 \\ D_1\left[U(g,t)-U^{i(1)}(g,t)\right]\Big|_{g\in\Gamma_1}=0, \quad D_2\left[U(g,t)\right]\Big|_{g\in\Gamma_2}=0, \quad t\geq 0, \end{cases} \tag{1}$$

where $\vec{E}=\{E_\rho,E_\varphi,E_z\}$ and $\vec{H}=\{H_\rho,H_\varphi,H_z\}$ are the electric and magnetic field vectors; $U(g,t)=E_\phi(g,t)$ for TE_{0n}-waves and $U(g,t)=H_\phi(g,t)$ for TM_{0n}-waves (Sirenko et al., 2007). The SI system of units is used. The variable t which being the product of the real time by the velocity of light in free space has the dimension of length. The operators D_1, D_2 will be described in Section 2 and provide an ideal model for fields emitted and absorbed by the waveguides.

The domain of analysis Ω is the part of the half-plane $\phi=\pi/2$ bounded by the contours Σ_ϕ together with the artificial boundaries Γ_j (input and output ports) in the virtual waveguides Ω_j, $j=1,2$. The regions $\Omega_{\text{int}}=\{g=\{r,\vartheta\}\in\Omega:\ r<L\}$ and Ω_{ext} (free space), such that $\Omega=\Omega_{\text{int}}\cup\Omega_{\text{ext}}\cup\Gamma$, are separated by the virtual boundary $\Gamma=\{g=\{r,\vartheta\}\in\Omega:\ r=L\}$.

The functions $F(g,t)$, $\varphi(g)$, $\psi(g)$, $\sigma(g)$, and $\varepsilon(g)-1$ which are finite in the closure $\bar{\Omega}$ of Ω are supposed to satisfy the hypotheses of the theorem on the unique solvability of problem (1) in the Sobolev space $\mathbf{W}_2^1(\Omega^T)$, $\Omega^T=\Omega\times(0;T)$ where $T<\infty$ is the observation time (Ladyzhenskaya, 1985). The 'current' and 'instantaneous' sources given by the functions $F(g,t)$ and $\varphi(g)$, $\psi(g)$ as well as all scattering elements given by the functions $\varepsilon(g)$, $\sigma(g)$ and by the contours Σ_ϕ and $\Sigma_\phi^{\varepsilon,\sigma}$ are located in the region Ω_{int}. In axially-symmetrical problems, at points g such that $\rho=0$, only H_z or E_z fields components are nonzero. Hence it follows that $U(0,z,t)=0$; $|z|<\infty$, $t\geq 0$ in (1).

3. Exact absorbing conditions for virtual boundaries in input-output waveguides

Equations

$$D_1\left[U(g,t)-U^{i(1)}(g,t)\right]\Big|_{g\in\Gamma_1}=0,\quad D_2\left[U(g,t)\right]\Big|_{g\in\Gamma_2}=0,\quad t\geq 0. \tag{2}$$

in (1) give the exact absorbing conditions for the outgoing pulsed waves $U^{s(1)}(g,t)=U(g,t)-U^{i(1)}(g,t)$ and $U^{s(2)}(g,t)=U(g,t)$ traveling into the virtual waveguides Ω_1 and Ω_2, respectively (Sirenko et al., 2007). $U^{i(1)}(g,t)$ is the pulsed wave that excites the axially-symmetrical structure from the circular or coaxial circular waveguide Ω_1. It is assumed that by the time $t=0$ this wave has not yet reached the boundary Γ_1.

By using conditions (2), we simplify substantially the model simulating an actual electrodynamic structure: the Ω_j-domains are excluded from consideration while the operators D_j describe wave transformation on the boundaries Γ_j that separate regular feeding waveguides from the radiating unit. The operators D_j are constructed such that a wave incident on Γ_j from the region Ω_{int} passes into the virtual domain Ω_j as if into a regular waveguide – without deformations or reflections. In other words, it is absorbed completely by the boundary Γ_j. Therefore, we call the boundary conditions (2) as well as the other conditions of this kind 'exact absorbing conditions'.

In the book (Sirenko et al., 2007), one can find six possible versions of the operators D_j for virtual boundaries in the cross-sections of circular or coaxial-circular waveguides. We pick out two of them (one for the nonlocal conditions and one for the local conditions) and, taking into consideration the location of the boundaries Γ_j in our problem (in the plane $z=-L_1$ for the boundary Γ_1 and in the plane $z=L_2$ for Γ_2) as well as the traveling direction for the waves outgoing through these boundaries (towards $z=-\infty$ for Γ_1 and towards $z=\infty$ for Γ_2), write (2) in the form:

$$U^{s(1)}(\rho,-L_1,t)=\sum_n\left\{\int_0^t J_0\left[\lambda_{n1}(t-\tau)\right]\left[\int_{b_1}^{a_1}\frac{\partial U^{s(1)}(\tilde\rho,z,\tau)}{\partial z}\bigg|_{z=-L_1}\mu_{n1}(\tilde\rho)\tilde\rho\, d\tilde\rho\right]d\tau\right\}\mu_{n1}(\rho),$$
$$b_1\leq\rho\leq a_1,\quad t\geq 0, \tag{3}$$

$$U(\rho,L_2,t)=-\sum_n\left\{\int_0^t J_0\left[\lambda_{n2}(t-\tau)\right]\left[\int_{b_2}^{a_2}\frac{\partial U(\tilde\rho,z,\tau)}{\partial z}\bigg|_{z=L_2}\mu_{n2}(\tilde\rho)\tilde\rho\, d\tilde\rho\right]d\tau\right\}\mu_{n2}(\rho),$$
$$b_2\leq\rho\leq a_2,\quad t\geq 0 \tag{4}$$

(nonlocal absorbing conditions) and

$$U^{s(1)}(\rho,-L_1,t)=\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial W_1(\rho,t,\varphi)}{\partial t}d\varphi,\quad t\geq 0,\quad b_1\leq\rho\leq a_1$$
$$\begin{cases}\left[\dfrac{\partial^2}{\partial t^2}-\sin^2\varphi\dfrac{\partial}{\partial\rho}\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\rho\right]W_1(\rho,t,\varphi)=\dfrac{\partial U^{s(1)}(\rho,z,t)}{\partial z}\bigg|_{z=-L_1}, & b_1\leq\rho\leq a_1,\quad t>0\\ W_1(\rho,0,\varphi)=\dfrac{\partial W_1(\rho,t,\varphi)}{\partial t}\bigg|_{t=0}=0, & b_1\leq\rho\leq a_1,\end{cases} \tag{5}$$

$$U(\rho,L_2,t)=\frac{2}{\pi}\int_0^{\pi/2}\frac{\partial W_2(\rho,t,\varphi)}{\partial t}d\varphi,\quad t\geq 0,\quad b_2\leq\rho\leq a_2$$

$$\begin{cases}\left[\frac{\partial^2}{\partial t^2}-\sin^2\varphi\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\right]W_2(\rho,t,\varphi)=-\left.\frac{\partial U(\rho,z,t)}{\partial z}\right|_{z=L_2}, & b_2\leq\rho\leq a_2,\quad t>0\\ W_2(\rho,0,\varphi)=\left.\frac{\partial W_2(\rho,t,\varphi)}{\partial t}\right|_{t=0}=0, & b_2\leq\rho\leq a_2\end{cases}\qquad(6)$$

(local absorbing conditions). The initial boundary-value problems involved in (5) and (6) with respect to the auxiliary functions $W_j(\rho,t,\varphi)$ must be supplemented with the following boundary conditions for all times $t\geq 0$:

$$\begin{cases}W_j(0,t,\varphi)=W_j(a_j,t,\varphi)=0 & (\mathrm{TE}_{0n}\text{-waves})\\ W_j(0,t,\varphi)=\left.\frac{\partial(\rho W_j(\rho,t,\varphi))}{\partial\rho}\right|_{\rho=a_j}=0 & (\mathrm{TM}_{0n}\text{-waves})\end{cases}\qquad(7)$$

(on the boundaries $\rho=0$ and $\rho=a_j$ of the region Ω_j for a circular waveguide) and

$$\begin{cases}W_j(b_j,t,\varphi)=W_j(a_j,t,\varphi)=0 & (\mathrm{TE}_{0n}\text{-waves})\\ \left.\frac{\partial(\rho W_j(\rho,t,\varphi))}{\partial\rho}\right|_{\rho=b_j}=\left.\frac{\partial(\rho W_j(\rho,t,\varphi))}{\partial\rho}\right|_{\rho=a_j}=0 & (\mathrm{TM}_{0n}\text{-waves})\end{cases}\qquad(8)$$

(on the boundaries $\rho=b_j$ and $\rho=a_j$ of the region Ω_j for a coaxial waveguide).
In (3) to (8) the following designations are used: $J_0(x)$ is the Bessel function, a_j and b_j are the radii of the waveguide Ω_j and of its inner conductor respectively (evidently, $b_j\neq 0$ if only Ω_j is a coaxial waveguide), $\{\mu_{nj}(\rho)\}$ and $\{\lambda_{nj}\}$ are the sets of transverse functions and transverse eigenvalues for the waveguide Ω_j .
Analytical representations for $\mu_{nj}(\rho)$ and λ_{nj} are well-known and for TE_{0n} -waves take the form:

$$\begin{cases}\mu_{nj}(\rho)=J_1(\lambda_{nj}\rho)\sqrt{2}\left[a_jJ_0(\lambda_{nj}a_j)\right]^{-1},\quad n=1,2,...,\\ \qquad\rho<a_j\quad\text{(circular waveguide)}\\ \lambda_{nj}>0\quad\text{are the roots of the equation}\quad J_1(\lambda a_j)=0,\end{cases}\qquad(9)$$

$$\begin{cases}\mu_{nj}(\rho)=G_1(\lambda_{nj},\rho)\sqrt{2}\left[a_j^2G_0^2(\lambda_{nj},a_j)-b_j^2G_0^2(\lambda_{nj},b_j)\right]^{-1/2},\quad n=1,2,...,\\ \qquad b_j<\rho<a_j\quad\text{(coaxial waveguide)}\\ \lambda_{nj}>0\quad\text{are the roots of the equation}\quad G_1(\lambda,a_j)=0\\ \qquad G_q(\lambda,\rho)=J_q(\lambda\rho)N_1(\lambda b_j)-N_q(\lambda\rho)J_1(\lambda b_j).\end{cases}\qquad(10)$$

For TM_{0n} -waves we have:

$$\begin{cases} \mu_{nj}(\rho) = J_1(\lambda_{nj}\rho)\sqrt{2}\left[a_j J_1(\lambda_{nj}a_j)\right]^{-1}, \quad n = 1,2,\ldots, \\ \qquad \rho < a_j \quad \text{(circular waveguide)} \\ \lambda_{nj} > 0 \quad \text{are the roots of the equation} \quad J_0(\lambda a_j) = 0, \end{cases} \tag{11}$$

$$\begin{cases} \mu_{nj}(\rho) = \tilde{G}_1(\lambda_{nj},\rho)\sqrt{2}\left[a_j^{\,2}\tilde{G}_1^2(\lambda_{nj},a_j) - b_j^{\,2}\tilde{G}_1^2(\lambda_{nj},b_j)\right]^{-1/2}, \quad n = 1,2,\ldots \\ \qquad \mu_0(\rho) = \left[\rho\sqrt{\ln(a_j/b_j)}\right]^{-1}, \quad b_j < \rho < a_j \quad \text{(coaxial waveguide)} \\ \lambda_{nj} > 0 \quad (n = 1,2,\ldots) \quad \text{are the roots of the equation} \quad \tilde{G}_0(\lambda,b_j) = 0, \quad \lambda_{0j} = 0 \\ \qquad \tilde{G}_q(\lambda,\rho) = J_q(\lambda\rho)N_0(\lambda a_j) - N_q(\lambda\rho)J_0(\lambda a_j). \end{cases} \tag{12}$$

Here $N_q(x)$ are the Neumann functions. The basis functions $\mu_{nj}(\rho)$ satisfy boundary conditions at the ends of the appropriate intervals ($\rho < a_j$ or $b_j < \rho < a_j$) and the following equalities hold

$$\int_0^{a_j} \mu_{nj}(\rho)\mu_{mj}(\rho)\rho d\rho = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \quad \text{or} \quad \int_{b_j}^{a_j} \mu_{nj}(\rho)\mu_{mj}(\rho)\rho d\rho = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \tag{13}$$

in the circular or coaxial waveguide, respectively.

4. Exact radiation conditions for outgoing spherical waves and exact absorbing conditions for the artificial boundary in free space

When constructing the exact absorbing condition for the wave $U(g,t)$ crossing the artificial spherical boundary Γ, we will follow the sequence of transformations widely used in the theory of hyperbolic equations (e.g., Borisov, 1996) – incomplete separation of variables in initial boundary-value problems for telegraph or wave equations, integral transformations in the problems for one-dimensional Klein-Gordon equations, solution of the auxiliary boundary-value problems for ordinary differential equations, and inverse integral transforms.

In the domain $\Omega_{\text{ext}} = \Omega \setminus (\Omega_{\text{int}} \cup \Gamma)$, where the field $U(g,t)$ propagates freely up to infinity as $t \to \infty$, the 2-D initial boundary-value problem (1) in spherical coordinates takes the form

$$\begin{cases} \left[-\dfrac{\partial^2}{\partial t^2} + \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2}r + \dfrac{1}{r^2}\dfrac{\partial}{\partial \vartheta}\left(\dfrac{1}{\sin\vartheta}\dfrac{\partial}{\partial \vartheta}\sin\vartheta\right)\right]U(g,t) = 0, \quad t > 0, \quad g = \{r,\vartheta\} \in \Omega_{\text{ext}} \\ U(g,t)\big|_{t=0} = 0, \quad \dfrac{\partial}{\partial t}U(g,t)\Big|_{t=0} = 0, \quad g \in \bar{\Omega}_{\text{ext}} \\ U(r,0,t) = U(r,\pi,t) = 0, \quad r \geq L, \quad t \geq 0. \end{cases} \tag{14}$$

Let us represent the solution $U(r,\vartheta,t)$ as $U(r,\vartheta,t)=u(r,t)\mu(\vartheta)$. Separation of variables in (14) results in a homogeneous Sturm-Liouville problem with respect to the function $\tilde{\mu}(\cos\vartheta)=\mu(\vartheta)$

$$\begin{cases} \left[\frac{d^2}{d\vartheta^2}+\operatorname{ctg}\vartheta\frac{d}{d\vartheta}-\frac{1}{\sin^2\vartheta}+\lambda^2\right]\tilde{\mu}(\cos\vartheta)=0, \quad 0<\vartheta<\pi \\ \tilde{\mu}(\cos\vartheta)\big|_{\vartheta=0,\pi}=0 \end{cases} \tag{15}$$

and the following initial boundary-value problem for $u(r,t)$:

$$\begin{cases} \left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}-\frac{\lambda^2}{r^2}\right]r\,u(r,t)=0, \quad r\geq L, \quad t>0 \\ u(r,0)=\frac{\partial}{\partial t}u(r,t)\Big|_{t=0}=0, \quad r\geq L. \end{cases} \tag{16}$$

Let us solve the Sturm-Liouville problem (15) with respect to $\tilde{\mu}(\cos\vartheta)$ and λ. Change of variables $x=\cos\vartheta$, $\tilde{\mu}(x)=\tilde{\mu}(\cos\vartheta)$ yields the following boundary-value problem for $\tilde{\mu}(x)$:

$$\begin{cases} \left[\left(1-x^2\right)\frac{d^2}{dx^2}-2x\frac{d}{dx}+\left(\lambda^2-\frac{1}{1-x^2}\right)\right]\tilde{\mu}(x)=0, \quad |x|<1 \\ \tilde{\mu}(-1)=\tilde{\mu}(1)=0. \end{cases} \tag{17}$$

With $\lambda^2=\lambda_n^2=n(n+1)$ for each $n=1,2,3,...$ equation (17) has two nontrivial linearly independent solutions in the form of the associated Legendre functions $P_n^1(x)$ and $Q_n^1(x)$. Taking into account the behavior of these functions in the vicinity of their singular points $x=\pm 1$ (Bateman & Erdelyi, 1953), we obtain

$$\tilde{\mu}_n(\cos\vartheta)=\sqrt{(2n+1)/(2n(n+1))}P_n^1(\cos\vartheta). \tag{18}$$

Here $\left\{\tilde{\mu}_n(\cos\vartheta)\right\}_{n=1,2,...}$ is a complete orthonormal (with weight function $\sin\vartheta$) system of functions in the space $\mathbf{L}_2\left[(0<\vartheta<\pi)\right]$ and provides nontrivial solutions to (15). Therefore, the solution of initial boundary-value problem (14) can be represented as

$$U(r,\vartheta,t)=\sum_{n=1}^{\infty}u_n(r,t)\tilde{\mu}_n(\cos\vartheta), \quad u_n(r,t)=\int_0^{\pi}U(r,\vartheta,t)\tilde{\mu}_n(\cos\vartheta)\sin\vartheta d\vartheta, \tag{19}$$

where the space-time amplitudes $u_n(r,t)$ are the solutions to problems (16) for $\lambda^2=\lambda_n^2$.
Our goal now is to derive the exact radiation conditions for space-time amplitudes $u_n(r,t)$ of the outgoing wave (19). By defining $w_n(r,t)=ru_n(r,t)$ and taking into account that $\lambda_n^2=n(n+1)$, we rewrite equation (16) as

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}-\frac{n(n+1)}{r^2}\right]w_n(r,t)=0, \quad r\geq L, \quad t>0. \tag{20}$$

Now subject it to the integral transform

$$\tilde{f}(\omega)=\int_L^{\infty} f(r)Z_\gamma(\omega,r)\,dr, \qquad \omega\geq 0, \tag{21}$$

where the kernel $Z_\gamma(\omega,r)=r^a\left[\alpha(\omega)J_\gamma(\omega r)+\beta(\omega)N_\gamma(\omega r)\right]$ satisfies the equation (Korn & Korn, 1961)

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1-2a}{r}\frac{\partial}{\partial r}+\omega^2+\frac{a^2-\gamma^2}{r^2}\right]Z_\gamma(\omega,r)=0. \tag{22}$$

Here $\alpha(\omega)$, $\beta(\omega)$ are arbitrary functions independent of r, and a is a fixed real constant. Applying to (20) the transform (21) with $a=1/2$ and $\gamma=n+1/2$, we arrive at

$$\int_L^{\infty}\left[-\frac{\partial^2}{\partial t^2}-\omega^2\right]w_n(r,t)Z_\gamma(\omega,r)\,dr+Z_\gamma(\omega,r)\frac{\partial w_n(r,t)}{\partial r}\bigg|_L^{\infty}-w_n(r,t)\frac{\partial Z_\gamma(\omega,r)}{\partial r}\bigg|_L^{\infty}=0. \tag{23}$$

Since the 'signal' $w_n(r,t)$ propagates with a finite velocity, for any t we can always point a distance r such that the signal has not yet reached it, that is, for these t and r we have $w_n(r,t)\equiv 0$. Then we can rewrite equation (23) in the form

$$\int_L^{\infty}\left[-\frac{\partial^2}{\partial t^2}-\omega^2\right]w_n(r,t)Z_\gamma(\omega,r)\,dr-Z_\gamma(\omega,L)\frac{\partial w_n(r,t)}{\partial r}\bigg|_{r=L}+w_n(L,t)\frac{\partial Z_\gamma(\omega,r)}{\partial r}\bigg|_{r=L}=0. \tag{24}$$

From (24) the simple differential equation for the transforms $\tilde{w}_n(\omega,t)$ of the functions $w_n(r,t)$ follows:

$$\left[\frac{\partial^2}{\partial t^2}+\omega^2\right]\tilde{w}_n(\omega,t)=w_n(L,t)\frac{\partial Z_\gamma(\omega,r)}{\partial r}\bigg|_{r=L}-Z_\gamma(\omega,L)\frac{\partial w_n(r,t)}{\partial r}\bigg|_{r=L}. \tag{25}$$

In this equation, the values $\alpha(\omega)$ and $\beta(\omega)$ entering into $Z_\gamma(\omega,r)$ are not defined yet. With $\alpha(\omega)=1$ and $\beta(\omega)=0$, we have

$$Z_\gamma(\omega,r)=\sqrt{r}J_\gamma(\omega r) \tag{26}$$

and

$$\tilde{f}(\omega)=\int_L^{\infty} f(r)\sqrt{r}J_\gamma(\omega r)\,dr. \tag{27}$$

The last integral is the Hankel transform (Korn & Korn, 1961), which is inverse to itself, and

$$\tilde{f}(\omega)\sqrt{\omega} = \int_0^\infty \left[f(r)\chi(r-L) \right] \sqrt{r\omega} J_\gamma(\omega r)\, dr,$$
$$f(r)\chi(r-L) = \int_0^\infty \left[\tilde{f}(\omega)\sqrt{\omega} \right] \sqrt{r\omega} J_\gamma(\omega r)\, d\omega. \tag{28}$$

By χ we denote the Heaviside step-function

$$\chi(r) = \begin{cases} 0 & \text{for} \quad r < 0 \\ 1 & \text{for} \quad r \geq 0 \end{cases}. \tag{29}$$

Taking into account (26), equation (25) can be rewritten in the form

$$\left[\frac{\partial^2}{\partial t^2} + \omega^2 \right] \tilde{w}_n(\omega, t) = g(\omega, t), \tag{30}$$

where

$$g(\omega,t) = w_n(L,t) \left[\frac{1}{2\sqrt{L}} J_\gamma(\omega L) + \omega\sqrt{L} J_\gamma'(\omega L) \right] - \sqrt{L} J_\gamma(\omega L) \left. \frac{\partial w_n(r,t)}{\partial r} \right|_{r=L}, \tag{31}$$

and the symbol $'$ denotes derivatives with respect to the whole argument ωL.
If G is a fundamental solution of the operator $B[G]$ (i.e., $B\left[G(t)\right] = \delta(t)$, where $\delta(t)$ is the Dirac delta function), then the solution to the equation $B\left[U(t)\right] = g(t)$ can be written as a convolution $U = (G * g)$ (Vladimirov, 1971). For $\left[\partial^2/\partial t^2 + \omega^2\right] G(t) = \delta(t)$ we have $G(t) = \chi(t)\omega^{-1}\sin\omega t$, and then

$$\tilde{w}_n(\omega,t) = \int_0^\infty G(\omega, t-\tau) g(\omega,\tau)\, d\tau = \frac{1}{2\omega\sqrt{L}} J_\gamma(\omega L) \int_0^t w_n(L,\tau) \sin\left[\omega(t-\tau)\right] d\tau +$$
$$+ \sqrt{L} J_\gamma'(\omega L) \int_0^t w_n(L,\tau) \sin\left[\omega(t-\tau)\right] d\tau - \frac{\sqrt{L}}{\omega} J_\gamma(\omega L) \int_0^t \left. \frac{\partial w_n(r,\tau)}{\partial r} \right|_{r=L} \sin\left[\omega(t-\tau)\right] d\tau. \tag{32}$$

Applying the inverse transform (28) to equation (32), we can write

$$w_n(r,t)\chi(r-L) = \int_0^\infty \tilde{w}_n(\omega,t)\, \omega\sqrt{r} J_\gamma(\omega r)\, d\omega =$$
$$= \int_0^t \int_0^\infty J_\gamma(\omega r) J_\gamma(\omega L) \sin\left[\omega(t-\tau)\right] d\omega \sqrt{r} \left[\frac{1}{2\sqrt{L}} w_n(L,\tau) - \sqrt{L} \left. \frac{\partial w_n(r,\tau)}{\partial r} \right|_{r=L} \right] d\tau + \tag{33}$$
$$+ \int_0^t \int_0^\infty \omega J_\gamma(\omega r) J_\gamma'(\omega L) \sin\left[\omega(t-\tau)\right] d\omega \sqrt{rL}\, w_n(L,\tau)\, d\tau.$$

Let us denote

$$I_1 = \int_0^\infty J_\gamma(\omega r) J_\gamma(\omega L) \sin\left[\omega(t-\tau)\right] d\omega \quad \text{and} \quad I_2 = \int_0^\infty \omega J_\gamma(\omega r) J_\gamma'(\omega L) \sin\left[\omega(t-\tau)\right] d\omega. \tag{34}$$

Then from (Gradshteyn & Ryzhik, 2000) we have for $r > L > 0$

$$I_1 = \begin{cases} 0, & 0 < t-\tau < r-L \\ \dfrac{1}{2\sqrt{rL}} P_{\gamma-1/2}\left(\dfrac{r^2+L^2-(t-\tau)^2}{2rL}\right), & r-L < t-\tau < r+L \\ -\dfrac{\cos\gamma\pi}{\pi\sqrt{rL}} Q_{\gamma-1/2}\left(-\dfrac{r^2+L^2-(t-\tau)^2}{2rL}\right), & t-\tau > r+L, \end{cases} \tag{35}$$

where $P_\gamma(x)$ and $Q_\gamma(x)$ are the Legendre functions of the first and second kind, respectively. For $\gamma = n+1/2$, we can rewrite this formula as

$$. I_1 = \frac{1}{2\sqrt{rL}} P_n(q) \chi\left[(t-\tau)-(r-L)\right]\chi\left[(r+L)-(t-\tau)\right], \quad t-\tau > 0, . \tag{36}$$

where $q = \left[r^2 + L^2 - (t-\tau)^2\right] / 2rL$ and $P_n(q)$ denotes a Legendre polynomial. Considering that

$$I_2 = \frac{\partial}{\partial L} I_1, \qquad \frac{\partial P_n(q)}{\partial q} = \frac{1}{\sqrt{1-q^2}} P_n^1(q) \tag{37}$$

(Janke et al., 1960), and $d\chi(x)/dx = \delta(x)$, we can derive

$$\begin{aligned} I_2 = {} & \frac{1}{2\sqrt{rL}} \chi\left[(t-\tau)-(r-L)\right]\chi\left[(r+L)-(t-\tau)\right]\left[-\frac{P_n(q)}{2L} + \frac{P_n^1(q)}{\sqrt{1-q^2}}\left(\frac{1}{r}-\frac{q}{L}\right)\right] + \\ & + \frac{1}{2\sqrt{rL}} P_n(q)\left\{\delta(t-\tau-r+L) + \delta(r+L-t+\tau)\right\}. \end{aligned} \tag{38}$$

Finally, taking into account the relation $w_n(r,t) = r u_n(r,t)$, we have from (33)

$$\begin{aligned} u_n(r,t) = {} & \frac{L}{2r}\left\{\int_{t-(r+L)}^{t-(r-L)}\left[\left(\frac{L-rq}{rL\sqrt{1-q^2}} P_n^1(q) - \frac{1}{L}P_n(q)\right) u_n(L,\tau) - P_n(q)\left.\frac{\partial u_n(r,\tau)}{\partial r}\right|_{r=L}\right] d\tau + \right. \\ & \left. + u_n\left(L, t-(r-L)\right) + (-1)^n u_n\left(L, t-(r+L)\right)\right\}, \qquad r > L. \end{aligned} \tag{39}$$

By using (19), we arrive at the desired radiation condition:

$$\begin{aligned} U(r,\vartheta,t) = {} & \frac{L}{2r}\sum_{n=1}^{\infty}\left\{\int_{t-(r+L)}^{t-(r-L)}\left[\left(\frac{L-rq}{rL\sqrt{1-q^2}} P_n^1(q) - \frac{1}{L}P_n(q)\right)\int_0^{\pi} U(L,\vartheta_1,\tau)\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1 - \right.\right. \\ & \left. - P_n(q)\int_0^{\pi}\left.\frac{\partial U(r,\vartheta_1,\tau)}{\partial r}\right|_{r=L}\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\right] d\tau + \end{aligned} \tag{40}$$

$$+\int_0^{\pi}\Big[U\big(L,\vartheta_1,t-(r-L)\big)+(-1)^n U\big(L,\vartheta_1,t-(r+L)\big)\Big]\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\Bigg\}\times$$
$$\times\tilde{\mu}_n(\cos\vartheta),\quad r>L,\quad 0\le\vartheta\le\pi .$$

By passing to the limit $r\to L$ in (40), we obtain

$$U(L,\vartheta,t)=\frac{1}{2}\sum_{n=1}^{\infty}\Bigg\{\int_{t-2L}^{t}\Bigg[\Bigg(\frac{t-\tau}{L\sqrt{4L^2-(t-\tau)^2}}P_n^1\bigg(1-\frac{(t-\tau)^2}{2L^2}\bigg)-\frac{1}{L}P_n\bigg(1-\frac{(t-\tau)^2}{2L^2}\bigg)\Bigg)\times$$
$$\times\int_0^{\pi}U(L,\vartheta_1,\tau)\,\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1-$$
$$-P_n\bigg(1-\frac{(t-\tau)^2}{2L^2}\bigg)\int_0^{\pi}\frac{\partial U(r,\vartheta_1,\tau)}{\partial r}\bigg|_{r=L}\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\Bigg]d\tau+ \tag{41}$$
$$+\int_0^{\pi}\Big[U(L,\vartheta_1,t)+(-1)^n U(L,\vartheta_1,t-2L)\Big]\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1\Bigg\}\tilde{\mu}_n(\cos\vartheta),$$
$$0\le\vartheta\le\pi.$$

Formula (41) represents the exact absorbing condition on the artificial boundary Γ. This condition is spoken of as exact because any outgoing wave described by the initial problem (1) satisfies this condition. Every outgoing wave $U(g,t)$ passes through the boundary Γ without distortions, as if it is absorbed by the domain Ω_{ext} or its boundary Γ. That is why this condition is said to be absorbing.

5. On the equivalence of the initial problem and the problem with a bounded domain of analysis

We have constructed the following closed initial boundary-value problem

$$\begin{cases}\left[-\varepsilon(g)\dfrac{\partial^2}{\partial t^2}-\sigma(g)\dfrac{\partial}{\partial t}+\dfrac{\partial^2}{\partial z^2}+\dfrac{\partial}{\partial\rho}\left(\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\rho\right)\right]U(g,t)=F(g,t),\quad t>0,\quad g\in\Omega_{\text{int}}\\ U(g,t)\big|_{t=0}=\varphi(g),\quad \dfrac{\partial}{\partial t}U(g,t)\bigg|_{t=0}=\psi(g),\quad g\in\bar{\Omega}_{\text{int}}\\ E_{tg}(p,t)\big|_{p=\{\rho,\phi,z\}\in\Sigma}=0,\quad U(0,z,t)=0,\quad |z|\le L,\quad t\ge 0\\ E_{tg}(p,t)\ \text{ and }\ H_{tg}(p,t)\ \text{ are continuous when crossing }\ \Sigma^{\varepsilon,\sigma}\\ D_1\Big[U(g,t)-U^{i(1)}(g,t)\Big]\Big|_{g\in\Gamma_1}=0,\quad D_2\big[U(g,t)\big]\big|_{g\in\Gamma_2}=0,\quad t\ge 0\\ D\big[U(g,t)\big]\big|_{g\in\Gamma}=0,\end{cases} \tag{42}$$

where the operator D is given by (41). It is equivalent to the open initial problem (1). This statement can be proved by following the technique developed in (Ladyzhenskaya, 1985).

The initial and the modified problems are equivalent if and only if any solution of the initial problem is a solution to problem (42) and at the same time, any solution of the modified problem is the solution to problem (1). (In the Ω_{ext}-domain, the solution to the modified problem is constructed with the help of (40).) The solution of the initial problem is unique and it is evidently the solution to the modified problem according construction. In this case, if the solution of (42) is unique, it will be a solution to (1). Assume that problem (42) has two different solutions $U_1(g,t)$ and $U_2(g,t)$. Then the function $u(g,t)=U_1(g,t)-U_2(g,t)$ is also the generalized solution to (42) for $F(g,t)=U^{i(1)}(g,t)=\varphi(g)=\psi(g)\equiv 0$. This means that for any function $\gamma(g,t)\in \mathbf{W}_2^1(\Omega^T)$ that is zero at $t=T$, the following equality holds:

$$\int_{\Omega_{\text{int}}^T}\left[\varepsilon\frac{\partial u}{\partial t}\frac{\partial\gamma}{\partial t}-\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho u\right)\frac{\partial(\rho\gamma)}{\partial\rho}-\frac{\partial u}{\partial z}\frac{\partial\gamma}{\partial z}-\sigma\frac{\partial u}{\partial t}\gamma\right]dgdt+ \\ +\int_{\Sigma_{\text{int}}^T}\left[\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho u\right)\gamma\cos(\vec{n},\vec{\rho})+\frac{\partial u}{\partial z}\gamma\cos(\vec{n},\vec{z})\right]dsdt=0. \tag{43}$$

Here, $\Omega_{\text{int}}^T=\Omega_{\text{int}}\times(0,T)$ and Σ_{int}^T are the space-time cylinder over the domain Ω_{int} and its lateral surface; $\cos(\vec{n},\vec{\rho})$ and $\cos(\vec{n},\vec{z})$ are the cosines of the angles between the outer normal $\vec{n}$ to the surface Σ_{int}^T and $\vec{\rho}$ - and $\vec{z}$ -axes, respectively; the element dg of the end surface of the cylinder Ω_{int}^T equals $\rho d\rho dz$.
By making the following suitable choice of function,

$$\gamma(g,t)=\begin{cases}\int_t^\tau u(g,\zeta)d\zeta & \text{for} \quad 0<t<\tau \\ 0 \quad \text{for} & \tau<t<T,\end{cases} \tag{44}$$

it is possible to show that every term in (43) is nonnegative (Mikhailov, 1976) and therefore $u(g,t)$ is equal to zero for all $g\in\Omega_{\text{int}}$ and $0<t<T$, which means that the solution to the problem (42) is unique. This proves the equivalency of the two problems.

6. Far-field zone problem. Extended and remote sources

As we have already mentioned, in contrast to approximate methods based on the use of the Absorbing Boundary Conditions or Perfectly Matched Layers, our approach to the effective truncation of the computational domain is rigorous, which is to say that the original open problem and the modified closed problem are equivalent. This allows one, in particular, to monitor a computational error and obtain reliable information about resonant wave scattering. It is noteworthy that within the limits of this rigorous approach we also obtain, without any additional effort, the solution to the far-field zone problem, namely, of finding the field $U(g,t)$ at arbitrary point in Ω_{ext} from the magnitudes of $U(g,t)$ on any arc $r=M\leq L$, $0\leq\vartheta\leq\pi$, lying entirely in Ω_{int} and retaining all characteristics of the arc Γ. Thus in the case considered here, equation (39) defines the diagonal operator $S_{L\to r}(t)$ such that it operates on the space of amplitudes $u(r,t)=\{u_n(r,t)\}$ of the outgoing wave (19) according the rule

$$u(r,t)=S_{L\to r}(t)\left[u(L,\tau),\left.\frac{\partial u(\tilde{r},\tau)}{\partial\tilde{r}}\right|_{\tilde{r}=L}\right]; \quad r>L, \quad t\geq\tau\geq 0, \tag{45}$$

and allows one to follow all variations of these amplitudes in an arbitrary region of Ω_{ext}. The operator

$$U(r,\vartheta,t)=T_{L\to r}(t)\left[U\left(L,\tilde{\vartheta},\tau\right),\left.\frac{\partial U\left(\tilde{r},\tilde{\vartheta},\tau\right)}{\partial\tilde{r}}\right|_{\tilde{r}=L}\right];\quad r>L,\quad 0\le\vartheta\le\pi,\quad t\ge\tau\ge 0,\quad 0\le\tilde{\vartheta}\le\pi, \tag{46}$$

given by (40) , in turn, enables the variations of the field $U(g,t)$, $g\in\Omega_{\text{ext}}$, to be followed.

It is obvious that the efficiency of the numerical algorithm based on (42) reduces if the support of the function $F(g,t)$ and/or the functions $\varphi(g)$ and $\psi(g)$ is extended substantially or removed far from the region where the scatterers are located. The arising problem (the far-field zone problem or the problem of extended and remote sources) can be resolved by the following straightforward way.

Let us consider the problem

$$\begin{cases}\left[-\varepsilon(g)\frac{\partial^2}{\partial t^2}-\sigma(g)\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}+\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\right)\right]U(g,t)=F(g,t)+\tilde{F}(g,t),\quad t>0,\quad g\in\Omega\\ \left.U(g,t)\right|_{t=0}=\varphi(g)+\tilde{\varphi}(g),\quad \left.\frac{\partial}{\partial t}U(g,t)\right|_{t=0}=\psi(g)+\tilde{\psi}(g),\quad g=\{\rho,z\}\in\bar{\Omega}\\ \left.E_{tg}(p,t)\right|_{p=\{\rho,\phi,z\}\in\Sigma}=0,\quad t\ge 0\\ E_{tg}(p,t)\ \text{ and }\ H_{tg}(p,t)\ \text{ are continuous when crossing }\ \Sigma^{\varepsilon,\sigma}\\ U(0,z,t)=0,\quad |z|<\infty,\quad t\ge 0\\ D_1\left[U(g,t)-U^{i(1)}(g,t)\right]\Big|_{g\in\Gamma_1}=0,\quad D_2\left[U(g,t)\right]\Big|_{g\in\Gamma_2}=0,\quad t\ge 0,\end{cases} \tag{47}$$

which differs from the problem (1) only in that the sources $\tilde{F}(g,t)$ and $\tilde{\varphi}(g)$, $\tilde{\psi}(g)$ are located out of the domain Ω_{int} enveloping all the scatterers (Fig. 1). The supports of the functions $\tilde{F}(g,t)$, $\tilde{\varphi}(g)$, and $\tilde{\psi}(g)$ can be arbitrary large (and even unbounded) and are located in Ω_{ext} at any finite distance from the domain Ω_{int}.

Let the relevant sources generate a field $U^i(g,t)$ in the half-plane $\Omega_0=\{g:\rho>0,|z|<\infty\}$. In other words, let the function $U^i(g,t)$ be a solution of the following Cauchy problem:

$$\begin{cases}\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\right)\right]U^i(g,t)=\tilde{F}(g,t),\quad t>0,\quad g\in\Omega_0\\ \left.U^i(g,t)\right|_{t=0}=\tilde{\varphi}(g),\quad \left.\frac{\partial}{\partial t}U^i(g,t)\right|_{t=0}=\tilde{\psi}(g),\quad g=\{\rho,z\}\in\bar{\Omega}_0\\ U^i(0,z,t)=0,\quad |z|<\infty,\quad t\ge 0.\end{cases} \tag{48}$$

It follows from (47), (48) that in the domain Ω_{ext} the function $U^s(g,t)=U(g,t)-U^i(g,t)$ satisfies the equations

$$\begin{cases} \left[\dfrac{\partial^2}{\partial t^2} + \dfrac{\partial^2}{\partial z^2} + \dfrac{\partial}{\partial \rho}\left(\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\rho \right) \right] U^s(g,t) = 0, \quad t > 0, \quad g \in \Omega_{\text{ext}} \\ U^s(g,t)\big|_{t=0} = 0, \quad \dfrac{\partial}{\partial t}U^s(g,t)\Big|_{t=0} = 0, \quad g = \{\rho, z\} \in \bar{\Omega}_{\text{ext}} \\ U^s(0,z,t) = 0, \quad |z| \geq L, \quad t \geq 0 \end{cases} \tag{49}$$

and determines there the pulsed electromagnetic wave crossing the artificial boundary Γ in one direction only, namely, from Ω_{int} into Ω_{ext}.

The problems (49) and (14) are qualitatively the same. Therefore, by repeating the transformations of Section 4, we obtain

$$\begin{aligned} U^s(L,\vartheta,t) = \frac{1}{2}\sum_{n=1}^{\infty}\Bigg\{ \int_{t-2L}^{t} \Bigg[\Bigg(\frac{t-\tau}{L\sqrt{4L^2 - (t-\tau)^2}} P_n^1\left(1 - \frac{(t-\tau)^2}{2L^2} \right) - \frac{1}{L}P_n\left(1 - \frac{(t-\tau)^2}{2L^2} \right) \Bigg) \times \\ \times \int_0^{\pi} U^s(L,\vartheta_1,\tau)\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1 - \\ - P_n\left(1 - \frac{(t-\tau)^2}{2L^2} \right) \int_0^{\pi} \frac{\partial U^s(r,\vartheta_1,\tau)}{\partial r}\Bigg|_{r=L} \tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1 \Bigg] d\tau + \\ + \int_0^{\pi}\left[U^s(L,\vartheta_1,t) + (-1)^n U^s(L,\vartheta_1,t-2L) \right]\tilde{\mu}_n(\cos\vartheta_1)\sin\vartheta_1 d\vartheta_1 \Bigg\} \tilde{\mu}_n(\cos\vartheta), \; 0 \leq \vartheta \leq \pi, \end{aligned} \tag{50}$$

or, in the operator notations, $D\left[U(g,t) - U^i(g,t) \right]\Big|_{g\in\Gamma} = 0$, – the exact absorbing condition allowing one to replace open problem (47) with the equivalent closed problem

$$\begin{cases} \left[-\varepsilon(g)\dfrac{\partial^2}{\partial t^2} - \sigma(g)\dfrac{\partial}{\partial t} + \dfrac{\partial^2}{\partial z^2} + \dfrac{\partial}{\partial \rho}\left(\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\rho \right) \right] U(g,t) = F(g,t), \quad t > 0, \quad g \in \Omega_{\text{int}} \\ U(g,t)\big|_{t=0} = \varphi(g), \quad \dfrac{\partial}{\partial t}U(g,t)\Big|_{t=0} = \psi(g), \quad g \in \bar{\Omega}_{\text{int}} \\ E_{tg}(p,t)\big|_{p=\{\rho,\phi,z\}\in\Sigma} = 0, \quad U(0,z,t) = 0, \quad |z| \leq L, \quad t \geq 0 \\ E_{tg}(p,t) \text{ and } H_{tg}(p,t) \text{ are continuous when crossing } \Sigma^{\varepsilon,\sigma} \\ D_1\left[U(g,t) - U^{i(1)}(g,t) \right]\Big|_{g\in\Gamma_1} = 0, \quad D_2\left[U(g,t) \right]\Big|_{g\in\Gamma_2} = 0, \quad t \geq 0 \\ D\left[U(g,t) - U^i(g,t) \right]\Big|_{g\in\Gamma} = 0. \end{cases} \tag{51}$$

7. Determination of the incident fields

To implement the algorithms based on the solution of the closed problems (42), (51), the values of the functions $U^{i(1)}(g,t)$ and $U^{i}(g,t)$ as well as their normal derivatives on the boundaries Γ_1 and Γ are required (see formulas (3), (5), (50)). Let us start from the function $U^{i(1)}(g,t)$. In the feeding waveguide Ω_1, the field $U^{i(1)}(g,t)$ incoming on the boundary Γ_1 can be represented (Sirenko *et al.*, 2007) as

$$U^{i(1)}(g,t)=\sum_n U_n^{i(1)}(g,t)=\sum_n v_{n1}(z,t)\,\mu_{n1}(\rho);\quad b_1<\rho<a_1,\quad z\le -L_1. \tag{52}$$

Here (see also Section 3), $n=0,1,2,...$ only in the case of TM_{0n}-waves and only for a coaxial waveguide Ω_1. In all other cases $n=1,2,3,...$. On the boundary Γ_1, the wave $U^{i(1)}(g,t)$ can be given by a set of its amplitudes $\{v_{n1}(-L_1,t)\}_n$. The choice of the functions $v_{n1}(-L_1,t)$, which are nonzero on the finite interval $0<T_1\le t\le T_2<T$, is arbitrary to a large degree and depends generally upon the conditions of a numerical experiment. As for the set $\left\{\partial v_{n1}(z,t)/\partial z\big|_{z=-L_1}\right\}_n$, which determines the derivative of the function $U^{i(1)}(g,t)$ on Γ_1, it should be selected with consideration for the causality principle. Each pair $V_{n1}(\rho,t)=\left\{v_{n1}(-L_1,t)\mu_{n1}(\rho);\left(\partial v_{n1}(z,t)/\partial z\big|_{z=-L_1}\right)\mu_{n1}(\rho)\right\}$ is determined by the pulsed eigenmode $U_n^{i(1)}(g,t)$ propagating in the waveguide Ω_1 in the sense of increasing z. This condition is met if the functions $v_{n1}(-L_1,t)$ and $\partial v_{n1}(z,t)/\partial z\big|_{z=-L_1}$ are related by the following equation (Sirenko et al., 2007):

$$v_{n1}(-L_1,t)=-\int_0^t J_0\left[\lambda_{n1}(t-\tau)\right]\frac{\partial v_{n1}(z,\tau)}{\partial z}\bigg|_{z=-L_1}d\tau;\quad t\ge 0. \tag{53}$$

The function $U^{i}(g,t)$ generated by the sources $\tilde{F}(g,t)$, $\tilde{\varphi}(g)$, and $\tilde{\psi}(g)$ is the solution to the Cauchy problem (48). Let us separate the transverse variable ρ in this problem and represent its solution in the form (Korn & Korn, 1961):

$$U^{i}(\rho,z,t)=\int_0^\infty v_\lambda(z,t)J_1(\lambda\rho)d\lambda \tag{54}$$

$$\begin{aligned} v_\lambda(z,t)&=\int_0^\infty v_\mu(z,t)\delta(\mu-\lambda)d\mu=\lambda\int_0^\infty v_\mu(z,t)\left[\int_0^\infty J_1(\mu\rho)J_1(\lambda\rho)\rho d\rho\right]d\mu=\\ &=\lambda\int_0^\infty\left[\int_0^\infty v_\mu(z,t)J_1(\mu\rho)d\mu\right]J_1(\lambda\rho)\rho d\rho=\lambda\int_0^\infty U^{i}(\rho,z,t)J_1(\lambda\rho)\rho d\rho. \end{aligned} \tag{55}$$

In order to find the functions $v_\lambda(z,t)$, one has to invert the following Cauchy problems for one-dimensional Klein-Gordon equations:

$$\begin{cases}\left[-\dfrac{\partial^2}{\partial t^2}+\dfrac{\partial^2}{\partial z^2}-\lambda^2\right]v_\lambda(z,t)=F_\lambda(z,t),\quad t>0,\quad |z|<\infty\\ v_\lambda(z,t)\big|_{t=0}=\varphi_\lambda(z),\quad \dfrac{\partial}{\partial t}v_\lambda(z,t)\bigg|_{t=0}=\psi_\lambda(z),\quad |z|<\infty.\end{cases} \tag{56}$$

Here, $F_\lambda(z,t)$, $\varphi_\lambda(z)$ и $\psi_\lambda(z,t)$ are the amplitude coefficients in the integral presentations (54) for the functions $\tilde{F}(g,t)$, $\tilde{\varphi}(g)$, and $\tilde{\psi}(g)$.
Now, by extending the functions $F_\lambda(z,t)$ and $v_\lambda(z,t)$ with zero on the interval $t<0$, we pass on to a generalized version of problems (56) (Vladimirov, 1971)

$$B(\lambda)[v_\lambda] \equiv \left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial z^2}-\lambda^2\right]v_\lambda(z,t) = F_\lambda(z,t)-\delta^{(1)}(t)\varphi_\lambda(z)-\delta(t)\psi_\lambda(z) = f_\lambda(z,t), \quad |t|<\infty, \quad |z|<\infty \tag{57}$$

($\delta^{(1)}(t)$ is the generalized derivative of the function $\delta(t)$). Their solutions can be written by using the fundamental solution $G(z,t,\lambda) = (-1/2)\chi(t-|z|)J_0\left[\lambda\left(t^2-z^2\right)^{1/2}\right]$ of the operator $B(\lambda)$ as follows:

$$v_\lambda(z,t) = \left[G(z,t,\lambda)*f_\lambda(z,t)\right] = \int_{-\infty}^{\infty}\int_0^{\infty} G(z-\tilde{z},t-\tau,\lambda) f_\lambda(\tilde{z},\tau)\,d\tau\, d\tilde{z}. \tag{58}$$

Equations (54) and (58) completely determine the desired function $U^i(g,t)$.

8. Conclusion

In this paper, a problem of efficient truncation of the computational domain in finite-difference methods is discussed for axially-symmetrical open electrodynamic structures. The original problem describing electromagnetic wave scattering on a compact axially-symmetric structure with feeding waveguides is an initial boundary-value problem formulated in an unbounded domain. The exact absorbing conditions have been derived for a spherical artificial boundary enveloping all sources and scatterers in order to truncate the computational domain and replace the original open problem by an equivalent closed one. The constructed solution has been generalized to the case of extended and remote field sources. The analytical representation for the operators converting the near-zone fields into the far-zone fields has been also derived.
We would like to make the following observation about our approach.

- In our description, the waveguide Ω_1 serves as a feeding waveguide. However, both of the waveguides can be feeding or serve to withdraw the energy; also both of them may be absent in the structure.
- The choice of the parameters $\alpha(\omega)$ and $\beta(\omega)$ determining $Z_\gamma(\omega,r)$ (see Section 4) affects substantially the final analytical expression for the exact absorbing condition on the spherical boundary Γ. When constructing boundary conditions (41), (50), we assumed that $\alpha(\omega)=1$ and $\beta(\omega)=0$. In (Sirenko et al., 2007), for a similar situation, the exact absorbing conditions for outgoing pulsed waves were constructed with the assumption that $\alpha(\omega)=-N_\gamma(\omega L)$ and $\beta(\omega)=J_\gamma(\omega L)$. With such $\alpha(\omega)$ and $\beta(\omega)$, equation (21) is the Weber-Orr transform (Bateman & Erdelyi, 1953). However, the final formulas corresponding to (39), (40) for this case turn into identities as $r \to L$, which present a considerable challenge for using them as absorbing conditions. In addition, the analytical expressions with the use of Weber-Orr transform are rather complicated to implement numerically.

- The function $U^i(g,t)$ (see Section 7) can be found in spherical coordinates as well. In this situation, we arrive (see Section 4) at the expansions like (19) with the amplitude coefficients $v_n(r,t)$ determined by the Cauchy problems

$$\begin{cases} \left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}-\frac{n(n+1)}{r^2}\right] r\, v_n(r,t)=F_n(r,t), \quad r\geq 0, \quad t>0 \\ v_n(r,0)=\varphi_n(r), \quad \left.\frac{\partial}{\partial t}v_n(r,t)\right|_{t=0}=\psi_n(r), \quad r\geq 0 \end{cases}, \qquad (59)$$

  where $F_n(r,t)$, $\varphi_n(r)$, and $\psi_n(r)$ are the amplitude coefficients for the functions $\tilde{F}(g,t)$, $\tilde{\varphi}(g)$, and $\tilde{\psi}(g)$.
- The standard discretization of the closed problems (42), (51) by the finite difference method using a uniform rectangular mesh attached to coordinates $g=\{\rho,z\}$ leads to explicit computational schemes with uniquely defined mesh functions $U(j,k,m)=U(\rho_j,z_k,t_m)$. The approximation error is $O(\bar{h}^2)$, where $\bar{h}$ is the mesh width in spatial coordinates, $\bar{l}=\bar{h}/2$ for $\theta=\max_{g\in\Omega_{\text{int}}}\left[\varepsilon(g)\right]<2$ or $\bar{l}<\bar{h}/2$ for $\theta\geq 2$ is the mesh width in time variable t; $\rho_j=j\bar{h}$, $z_k=k\bar{h}$, and $t_m=m\bar{l}$. The range of the integers $j=0,1,...,J$, $k=0,1,...K$, and $m=0,1,...M$ depends both on the size of the Ω_{int} domains and on the length of the interval $[0,T]$ of the observation time t. The condition providing uniform boundedness of the approximate solutions $U(j,k,m)$ with decreasing $\bar{h}$ and $\bar{l}$ is met (see, for example, formula (1.50) in (Sirenko *et al.*, 2007)). Hence the finite-difference computational schemes are stable, and the mesh functions $U(j,k,m)$ converge to the solutions $U(\rho_j,z_k,t_m)$ of the original problems (42), (51).

As opposed to the well-known approximate boundary conditions standardly utilized by finite-difference methods, the conditions derived in this paper are exact by construction and do not introduce an additional error into the finite-difference algorithm. This advantage is especially valuable in resonant situations, where numerical simulation requires large running time and the computational errors may grow unpredictably if an open problem is replaced by an insufficiently accurate closed problem.

9. References

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